

# WHAT IS VARIABLE BANDWIDTH?

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ABSTRACT. We propose a new notion of variable bandwidth that is based on the spectral subspaces of an elliptic operator  $A_p f = -\frac{d}{dx}(p(x)\frac{d}{dx})f$  where  $p > 0$  is a strictly positive function. Denote by  $c_\Lambda(A_p)$  the orthogonal projection of  $A_p$  corresponding to the spectrum of  $A_p$  in  $\Lambda \subset \mathbb{R}^+$ , the range of this projection is the space of functions of variable bandwidth with spectral set in  $\Lambda$ .

We will develop the basic theory of these function spaces. First, we derive (nonuniform) sampling theorems, second, we prove necessary density conditions in the style of Landau. Roughly, for a spectrum  $\Lambda = [0, \Omega]$  the main results say that, in a neighborhood of  $x \in \mathbb{R}$ , a function of variable bandwidth behaves like a bandlimited function with local bandwidth  $(\Omega/p(x))^{1/2}$ .

Although the formulation of the results is deceptively similar to the corresponding results for classical bandlimited functions, the methods of proof are much more involved. On the one hand, we use the oscillation method from sampling theory and frame theoretic methods, on the other hand, we need the precise spectral theory of Sturm-Liouville operators and the scattering theory of one-dimensional Schrödinger operators.

## 1. INTRODUCTION

A function  $f \in L^2(\mathbb{R})$  has bandwidth  $\Omega > 0$ , if its Fourier transform  $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi} dx$  vanishes outside the interval  $[-\Omega, \Omega]$ . The number  $\Omega$  is the maximal frequency contributing to  $f$  and is called the bandwidth of  $f$ . According to Shannon the bandwidth is an information-theoretic quantity and determines how many samples of a function  $f$  are required to determine  $f$  completely. Alternatively, the bandwidth indicates how much information can be transmitted through a communication channel.

In the context of time-frequency analysis it is perfectly plausible to assign different local bandwidths to different segments of a signal. This becomes even more obvious in the often cited metaphor of music: the highest frequency of musical piece is time-varying. However, a rigorous definition of variable bandwidth is difficult, perhaps even elusive, because bandwidth is global by definition and the assignment of a local bandwidth is in contradiction with the uncertainty principle.

So what is a function of variable bandwidth?

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Before attempting to give a precise definition, we need to single out the distinctive features of bandlimited functions. In our view the essence of bandwidth is encapsulated in three fundamental theorems about bandlimited functions:

- (i) the Shannon-Whittaker-Kotelnikov sampling theorem and its variations,
- (ii) the existence of a critical density in the style of Landau's necessary density conditions (a Nyquist rate in engineering terms), and
- (iii) some inherent analyticity, expressed by a Bernstein-type inequality and theorems in the style of Paley-Wiener.

Our objective is to introduce a concept of variable bandwidth that shares these fundamental properties (sampling theorems and density results) with classical bandlimited functions. Our starting point is the well known observation that bandlimited functions are contained in a spectral subspace of the differential operator  $-D^2 = -\frac{d^2}{dx^2}$ . It is diagonalized by the Fourier transform  $\mathcal{F}$  and  $\mathcal{F}D^2\mathcal{F}^{-1}f = \xi^2\hat{f}(\xi)$  is simply the operator of multiplication by  $\xi^2$ . For fixed  $\Omega > 0$  the spectral subspace corresponding to the spectral values  $0 \leq \xi^2 \leq \Omega$  is defined by the spectral projection  $c_{[0,\Omega]}(-D^2)$ , which is given by  $\hat{f} \mapsto c_{[0,\Omega]}(\xi^2)\hat{f}(\xi)$  in the Fourier domain. Consequently the spectral subspace  $c_{[0,\Omega]}(-D^2)L^2(\mathbb{R})$  is identical with the bandlimited functions of bandwidth  $\Omega^{1/2}$ .

Our idea is to replace the constant coefficient differential operator  $-D^2$  by an elliptic differential operator in divergence form

$$(1.1) \quad A_p = -D(p(x)D),$$

with a *bandwidth-parametrizing function*  $p > 0$ . As above, a space of variable bandwidth is given by a spectral subspace of the differential operator  $A_p$ . By imposing mild assumptions on  $p$  and choosing a suitable domain,  $A_p$  becomes a positive, unbounded, self-adjoint operator on  $L^2(\mathbb{R})$ . Its spectral representation enables us to make the following definition.

**Definition 1.1.** Let  $\Lambda \subseteq \mathbb{R}^+$  be a fixed Borel set with finite Lebesgue measure. A function is  $A_p$ -bandlimited with spectral set  $\Lambda$ , if  $f \in c_\Lambda(A_p)L^2(\mathbb{R})$ . The range of the spectral projection  $c_\Lambda(A_p)L^2(\mathbb{R})$  is called the Paley-Wiener space with respect to  $A_p$  and spectral set  $\Lambda$  and will be denoted by  $PW_\Lambda(A_p)$ . The quantity  $\Omega = \max\{\lambda \in \Lambda\}$  is the bandwidth of  $PW_\Lambda(A_p)$ . If  $\Lambda = [0, \Omega]$ , we will often speak of functions of variable bandwidth  $\Omega$ .

If  $p \equiv 1$  and  $A_p = -\frac{d^2}{dx^2}$ , then, as argued above,  $PW_{[0,\Omega]}(A_p)$  consists exactly of the classical bandlimited functions with bandwidth  $\Omega^{1/2}$ .

Our challenge is to convince the reader that Definition 1.1 is indeed a meaningful notion of variable bandwidth. We must interpret functions in  $PW_\Lambda(A_p)$  as functions of variable bandwidth and need to relate the parametrizing function  $p$  to a local bandwidth. Furthermore, we need to establish sampling and density theorems for  $PW_\Lambda(A_p)$  that depend to the bandwidth-parametrizing function  $p$ . As a guideline, we would expect that  $1/\sqrt{p(x)}$  determines the local bandwidth in a neighborhood of  $x$  and will enter the formulation of the basic results.

First, we show that functions of variable bandwidth admit sampling theorems.

**Theorem 1.2** (Sampling theorem for  $PW_\Lambda(A_p)$ ). *Fix  $\Lambda \subseteq \mathbb{R}^+$  compact and set  $\Omega = \max \Lambda$ . Assume that  $0 < c \leq p(x)$  for all  $x \in \mathbb{R}$ . Let  $X = (x_i)_{i \in \mathbb{Z}}$  be an increasing sequence with  $\lim_{i \rightarrow \pm\infty} x_i = \pm\infty$  and  $\inf_i (x_{i+1} - x_i) > 0$ . If*

$$(1.2) \quad \delta = \sup_{i \in \mathbb{Z}} \frac{x_{i+1} - x_i}{\inf_{x \in [x_i, x_{i+1}]} \sqrt{p(x)}} < \frac{\pi}{\Omega^{1/2}},$$

*then there exist  $A, B > 0$  such that, for all  $f \in PW_\Lambda(A_p)$ ,*

$$(1.3) \quad A\|f\|_2^2 \leq \sum_{i \in \mathbb{Z}} |f(x_i)|^2 \leq B\|f\|_2^2$$

Since  $PW_\Lambda(A_p)$  is a reproducing kernel Hilbert space, the sampling inequality (1.3) implies a variety of reconstruction algorithms. Following [12] we will formulate an iterative algorithm for the reconstruction of  $f \in PW_\Lambda(A_p)$  from the samples  $\{f(x_i) : i \in \mathbb{Z}\}$  with geometric convergence.

Theorem 1.2 supports our interpretation that  $p(x)^{-1/2}$  is a measure for the local bandwidth. If  $p$  is constant on an interval  $I$ ,  $p|_I = p_0$ , then the maximum gap condition (1.2) reads as  $x_{i+1} - x_i \leq \delta \sqrt{p_0} < \pi(p_0/\Omega)^{1/2}$  for  $x_i \in I$ . This is precisely the sufficient condition on the maximal gap that arises for bandlimited functions with bandwidth  $(\Omega/p_0)^{1/2}$ . In other words,  $f \in PW_{[0, \Omega]}(A_p)$  behaves like a  $(\Omega/p_0)^{1/2}$ -bandlimited function on  $I$ .

We remark that condition (1.2) is almost optimal; the constant  $\frac{\pi}{\Omega^{1/2}}$  in (1.2) cannot be improved. However, a weaker, qualitative version of this sampling theorem with a sufficiently small  $\delta$  in (1.2) can be derived from Pesenson's theory of abstract bandwidth [32, 34].

Our second main result is a necessary density condition for sampling in the style of Landau [26, 25]. For the formulation we need an adaptation of the Beurling density to variable bandwidth. As in (1.2) we impose a new measure or distance on  $\mathbb{R}$  determined by the bandwidth parametrization  $p$ , namely  $\mu_p(I) = \int_I p^{-1/2}(u) du$  and define the Beurling density of a set  $X \subseteq \mathbb{R}$  as

$$D_p^-(X) = \lim_{r \rightarrow \infty} \inf_{\mu_p(I)=r} \frac{\#\{X \cap I\} : I \subset \mathbb{R} \text{ closed interval}}{r}.$$

We write  $\Lambda^{1/2} = \{\omega \in \mathbb{R}^+ : \omega^2 \in \Lambda\}$  for the square root of a set and  $|\Lambda^{1/2}|$  for its Lebesgue measure.

**Theorem 1.3.** *Assume that  $p \in C^2$  and  $p$  is eventually constant, i.e., for some  $R > 0$  we have  $p(x) = p_-$  for  $x \leq -R$  and  $p(x) = p_+$  for  $x \geq R$ . Fix  $\Lambda \subseteq \mathbb{R}^+$  with finite (Lebesgue) measure. If  $X \subseteq \mathbb{R}$  is a separated set such that the sampling inequality*

$$A\|f\|_2^2 \leq \sum_{i \in I} |f(x_i)|^2 \leq B\|f\|_2^2$$

*holds for all  $f \in PW_\Lambda(A_p)$ , then  $D_p^-(X) \geq \frac{|\Lambda^{1/2}|}{\pi}$ .*

Theorem 1.3 is again consistent with our interpretation of  $PW_\Lambda(A_p)$  as a space of functions with variable bandwidth. If  $\Lambda = [0, \Omega]$  and  $p$  is constant on an interval

$I$ ,  $p|_I = p_0$ , then  $\mu_p(I) = |I|/\sqrt{p_0}$  and we obtain roughly

$$\#(X \cap I) \geq \frac{|\Lambda^{1/2}| |I|}{\pi \sqrt{p_0}} = \left( \frac{\Omega}{p_0} \right)^{1/2} \frac{|I|}{\pi}.$$

Comparing with Landau's classical result for bandlimited function, this is exactly the minimum number of samples in  $I$  required for a bandlimited function with bandwidth  $(\Omega/p_0)^{1/2}$ . Again,  $f \in PW_{[0,\Omega]}(A_p)$  behaves like a  $(\Omega/p_0)^{1/2}$ -bandlimited function on  $I$ .

The two main theorems demonstrate convincingly that the spectral subspaces  $PW_\Lambda(A_p)$  are indeed appropriate models for spaces of functions with variable bandwidth. The values  $p(x)^{-1/2}$  may be taken as a measure for the local bandwidth and enter significantly in the formulation of sampling and density theorems for these spaces.

The inherent analyticity properties of functions of variable bandwidth (item (iii) on our wishlist) follow from the theory of abstract bandwidth [32, 34] and will be discussed in Section 3.

**Methods.** The formulation of the main theorems looks like a small variation of the standard theorems for classical bandlimited functions with the parametrizing function  $p$  appearing in the appropriate places. The proofs of the above theorems, however, require input from two areas, namely the applied harmonic analysis of sampling theory and the detailed spectral analysis of Sturm-Liouville operators and Schrödinger operators. The methodical input from sampling theory is the oscillation method from [12] for the proof of Theorem 1.2, whereas the proof of Theorem 1.3 follows the outline of Nitzan and Olevski [30] in which a (discrete) frame of reproducing kernels is compared to a continuous resolution of the identity. The second methodical input is from the theory of Sturm-Liouville problems and of (one-dimensional) Schrödinger operators. To see why we need the extensive build-up of Sturm-Liouville theory, we recall that much of the theory of classical bandlimited functions is based on the Fourier transform and the explicit formula for the reproducing kernel  $k(x, y) = \frac{\sin(x-y)}{x-y}$  of the standard Paley-Wiener space. Our main effort is devoted to finding appropriate substitutes for these explicit expressions. On the one hand, we find these in the spectral theory of Sturm-Liouville operators. The detailed analysis of the spectral measure of  $A_p$  yields a representation of functions in  $PW_\Lambda(A_p)$  as

$$f(x) = \int_{\Lambda} F(\lambda) \cdot \Phi(\lambda, x) d\rho(\lambda),$$

where  $\Phi(\lambda, x) = (\Phi_1(\lambda, x), \Phi_2(\lambda, x))^T$  is a set of fundamental solutions of  $-(p\Phi')' = \lambda\Phi$ ,  $\rho$  is the  $2 \times 2$ -matrix-valued spectral measure, and  $F \in L^2(\Lambda, d\rho)$ . Though not as explicit as the Fourier transform, this spectral representation of functions of variable bandwidth will enable us to derive the essential properties of  $PW_\Lambda(A_p)$ .

For the proof of the density theorem we will switch to an equivalent Schrödinger equation. By applying a Liouville transform, the differential operator  $-Dp(x)D$  is unitarily equivalent to a one-dimensional Schrödinger operator  $B_p = -D^2 + q(x)$

where  $q$  is an explicit expression depending on the bandwidth parametrization  $p$  (see (6.6) for the precise formula). The advantage of the Schrödinger picture is that we can apply the scattering theory of the Schrödinger equation to obtain asymptotic estimates for the reproducing kernel of  $PW_\Lambda(B_q)$ . To appreciate the transition to the Schrödinger picture, the reader should at least check the remark after the proof of Lemma 6.9. 7.

The additional insight emerging from this approach is that sampling sets for  $PW_\Lambda(A_p)$  are obtained from sampling sets for the Paley-Wiener-space of the Schrödinger operator  $PW_\Lambda(B_q)$  (which is defined verbatim as in Definition 1.1) by means of time-warping with the Liouville transform. Note, however, that the concrete interpretation of variable bandwidth is lost in the Schrödinger picture.

Let us emphasize that at this time we want to focus on the proof of concept and to convince the reader that Definition 1.1 yields a meaningful and mathematically interesting notion of variable bandwidth. Therefore we develop the theory of variable bandwidth under rather benign assumptions on the bandwidth parametrization  $p$ . It is clear that on a technical level our results can and should be pushed much further by using more advanced aspects of Sturm-Liouville theory. Indeed, once the basic theory is established, we face a host of new, interesting, and non-trivial questions in sampling theory. See the last section for a look at ongoing work.

As the paper should be accessible for two different communities (applied harmonic analysis and spectral analysis), we have tried to work on a moderate technical level. This means in particular that we feel the need to summarize some well-known parts of the spectral theory to harmonic analysts.

**Related work and other notions of variable bandwidth.** In the literature one finds several approaches to variable bandwidth.

1. *Time-warping* [6, 20, 5, 37, 40]: Given a homeomorphism  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$  (a warping function), a function  $f$  possesses variable bandwidth with respect to  $\gamma$ , if  $f = g \circ \gamma$  for a bandlimited function  $g \in L^2(\mathbb{R})$  with  $\text{supp } \hat{g} \subseteq [-\Omega, \Omega]$ . The derivative  $1/\gamma'(\gamma^{-1}(x))$  of the warping function is interpreted as the local bandwidth of  $f$ . Although the sampling theory for time-warped functions is simple, time-warping is relevant in signal processing, and the estimate of suitable warping functions for given data is an important problem. Let us mention that time-warping functions can be understood as spectral subspaces of certain differential operators of order one (see Appendix B).

2. *Aceska and Feichtinger* [1, 2] have proposed a concept of variable bandwidth based on time-frequency methods, namely the truncation of the short time Fourier transform by means of a time-varying frequency cut-off. The resulting function spaces, however, coincide with the standard Bessel-Sobolev potential spaces endowed with an equivalent norm. Since these spaces do not admit a sampling theorem (nor a Nyquist density), they are not variable bandwidth spaces in our sense. The spaces of [1, 2] are rather spaces of locally variable smoothness.

3. *Kempf and his collaborators* [21, 22, 18] use a procedural concept of variable bandwidth, but (at least in the available literature) shy away from a formal definition. The parametrization of self-adjoint extensions of a differential operator

leads to a class of algorithms that reconstruct or interpolate a function from certain samples. The reconstructed function is said to have variable bandwidth.

4. *Abstract Paley-Wiener spaces* [32, 34, 45]: Perhaps closest to our approach is the work of Pesenson and Zayed on abstract bandlimitedness. Given an unbounded, self-adjoint operator on a Hilbert space  $H$ , the spectral subspaces  $c_\Lambda(A)H$  are considered abstract spaces of bandlimited vectors. If  $A$  is the Laplace-Beltrami operator on a manifold, then the corresponding Paley-Wiener spaces are concrete function spaces and admit sampling theorems. These results are merely qualitative and so far are not backed up by corresponding density results. Paley-Wiener spaces on manifolds play an important role in the construction and analysis of Besov spaces on various manifolds. See [11, 33, 7, 23] for this direction of research.

5. *Sampling theory associated with Sturm-Liouville problems* [44]: The generalizations of Kramer's sampling theorem with Sturm-Liouville theory aim at interpolation formulas and sampling theorems analogous to the cardinal series. However, the samples are taken on the spectral side and the conditions imposed on the Sturm-Liouville operator guarantee a discrete spectrum, in contrast to our set-up. Except for the use of Sturm-Liouville theory, we do not see any connection to our work.

**Organization.** The paper is organized as follows: In Section 2 we review the relevant aspects of the spectral theory of Sturm-Liouville operators. In Section 3 we collect the basic properties of functions of variable bandwidth. Section 4 treats the toy example of the discontinuous parametrizing function  $p(x) = p_-$  for  $x \leq 0$  and  $p(x) = p_+$  for  $x > 0$  and  $p_- \neq p_+$ . We will show that the corresponding Paley-Wiener space consists of functions with different bandwidths  $1/\sqrt{p_\pm}$  on the positive and negative half axis and then prove a Shannon-like sampling theorem (Thm. 4.1). This example is instructive because all objects (spectral measure, reproducing kernel) can be computed explicitly. In Section 5 we prove Theorem 1.2 and treat the stable reconstruction of a function of variable bandwidth from a sampling sequence. The approach is based on [12, 13]. In Section 6 we define a Beurling density adapted to the sampling geometry of the differential operator and prove necessary density conditions for stable sampling and interpolation in the style of Landau. The proof follows the outline of [30], the main technical work is to control the reproducing kernels and its oscillations, which is done in Section 7.

The appendices contain our remarks on time-warping and the explicit calculations needed for Section 4. Such material is usually left to the interested reader, but we prefer to include it, because we have struggled several times to reproduce our own calculations.

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## 2. SPECTRAL THEORY OF SOME SECOND ORDER DIFFERENTIAL OPERATORS

In this section we provide the necessary background from the spectral theory of second order differential operators, in particular, of Sturm-Liouville operators. Our standard references for the material are [39, 41, 42] and also [9].

We consider differential expressions of Sturm-Liouville type

$$(2.1) \quad \tau f = -(pf')' + qf$$

where  $p, q$  are measurable functions satisfying

$$(2.2) \quad p > 0 \text{ a.e.}, \text{ and } 1/p, q \in L^1_{loc}(\mathbb{R}).$$

The corresponding *maximal operator*  $A$  is defined by the choice of the domain

$$(2.3) \quad \begin{aligned} \mathcal{D}(A) &= \{f \in L^2(\mathbb{R}) : f, pf' \in AC_{loc}, \tau f \in L^2(\mathbb{R})\} \\ Af &= \tau f \quad \text{for } f \in \mathcal{D}(A_p). \end{aligned}$$

We will always impose the conditions (2.2) and (2.3) without further notice.

For the study of variable bandwidth we will focus on differential expressions in divergence form,

$$(2.4) \quad \tau_p f = -(pf')'$$

with the corresponding maximal operator  $A_p$ . For the density theorem we will also consider Schrödinger operators

$$(2.5) \quad \tilde{\tau}_q f = -f'' + qf$$

with corresponding maximal operator  $B_q$ . Under mild assumptions  $A_p$  and  $B_q$  are unitarily equivalent via the Liouville transform, see Section 6.2.

To cite the necessary results from the literature we need some more terminology and notation.

A solution  $\phi$  of  $(\tau - z)\phi = 0$ ,  $z \in \mathbb{C}$ , *lies left* in  $L^2(\mathbb{R})$ , if  $\phi \in L^2((-\infty, c))$  for some  $c \in \mathbb{R}$ , and *lies right* if  $\phi \in L^2((c, \infty))$  for some  $c \in \mathbb{R}$ . If for every  $z \notin \mathbb{R}$  there are two unique (up to a multiplicative constant) solutions of  $(\tau - z)\phi = 0$  that lie left respectively right in  $L^2(\mathbb{R})$  (jargon:  $\tau$  is *limit point* (LP) at  $\pm\infty$ ), then the corresponding maximal operator  $A$  is self-adjoint [42, 13.18, 13.19]. This is the only situation we treat in this text.

We cite a simple sufficient condition on  $p$  such that the maximal operator  $A_p$  corresponding to  $\tau_p$  is self-adjoint.

**Proposition 2.1.** *Let  $P(x) = \int_0^x p(u)^{-1} du$ . If  $P \notin L^2(\mathbb{R}^+)$  and  $P \notin L^2(\mathbb{R}^-)$ , then  $A_p$  is self-adjoint.*

For a proof see [42, 13.24] (or [41, Thm. 6.3]) in conjunction with [42, 13.8, 13.19].

*Remark.* Under minimal additional assumptions, one can deduce more information about the spectrum of  $A_p$ . For instance, if there exist constants  $C_1, C_2 > 0$  such



that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left| 1 - \frac{C_1}{p(u)} \right| du = 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^0 \left| 1 - \frac{C_2}{p(u)} \right| du = 0,$$

then  $\sigma(A_p) = \sigma_{ess}(A_p) = [0, \infty)$ , cf. [41, Thm. 15.1]. A sufficient condition for  $\sigma_{ac}(A_p) = \sigma_{ess}(A_p) = [0, \infty)$ ,  $\sigma_{sc} = \emptyset$  is given in [36, Thm 3.2].

In particular the conditions of Proposition 2.1 are satisfied for eventually constant functions  $p$ , i.e.,  $p$  satisfies

$$(MC) \quad p(x) = \begin{cases} p_-, & x < -R \\ p_+, & x > R \end{cases}$$

for an  $R > 0$ . We call this case the *model case*.

For a self-adjoint realization  $A$  of a differential expression  $\tau$  of Sturm-Liouville type the functional calculus can be described more explicitly than by the spectral theorem alone. Our reference for most of the following is mainly [42], and also [39, 9].

Let  $\rho$  be a positive semi-definite  $2 \times 2$  matrix-valued Borel measure (a *positive matrix measure*), and  $L^2(\mathbb{R}, d\rho)$  the completion of the space of simple  $\mathbb{C}^2$ -valued functions  $F, G$  with respect to the scalar product

$$(2.6) \quad \int_{\mathbb{R}} F(\lambda) \cdot \overline{G(\lambda)} d\rho(\lambda) = \int_{\mathbb{R}} \sum_{j,k=1}^2 F_j(\lambda) \overline{G_k(\lambda)} d\rho_{jk}(\lambda).$$

Note that the trace  $\mu = \text{Tr } \rho$  is a positive Borel measure and the components  $\rho_{jk}$  are absolutely continuous with respect to  $\mu$ . See [9, XIII.5] for basic properties of matrix measures.

**Theorem 2.2** ([42, 14.1, 14.3], [39, Lem. 9.28], [9, XIII.5]). *Assume that  $\tau$  is a differential expression of Sturm Liouville type in LP condition at  $\pm\infty$  and that  $A$  is the corresponding self-adjoint operator.*

*If  $\Phi(\lambda, x) = (\phi_1(\lambda, x), \phi_2(\lambda, x))$  is a fundamental system of solutions of  $(\tau - \lambda)\phi = 0$  that depends continuously on  $\lambda$ , then there exists a  $2 \times 2$  matrix measure  $\rho$ , such that the operator*

$$(2.7) \quad \mathcal{F}_A : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}, d\rho); \quad \mathcal{F}_A f(\lambda) = \int_{\mathbb{R}} f(x) \overline{\Phi(\lambda, x)} dx$$

*is unitary and diagonalizes  $A$ , i.e.,  $\mathcal{F}_A A \mathcal{F}_A^{-1} G(\lambda) = \lambda G(\lambda)$  for all  $G \in L^2(\mathbb{R}, d\rho)$ . The inverse has the form*

$$\mathcal{F}_A^{-1} G(x) = \int_{\mathbb{R}} G(\lambda) \cdot \Phi(\lambda, x) d\rho(\lambda).$$

*for  $G \in L^2(\mathbb{R}, d\rho)$ .*

*If  $g$  is a bounded Borel function on  $\mathbb{R}$  then, for every  $f \in L^2(\mathbb{R})$ ,*

$$(2.8) \quad g(A)f(x) = \int_{\mathbb{R}^+} g(\lambda) \mathcal{F}_A f(\lambda) \cdot \Phi(\lambda, x) d\rho(\lambda).$$



All integrals  $\int_{\mathbb{R}}$  have to be understood as  $\lim_{\substack{a \rightarrow -\infty \\ b \rightarrow \infty}} \int_a^b$  with convergence in  $L^2$ .

Note that the spectral projection  $c_{\Lambda}(A)$  is given by  $c_{\Lambda}(A)f(x) = \int_{\Lambda} \mathcal{F}_A f(\lambda) \cdot \Phi(\lambda, x) d\rho(\lambda)$ .

$\mathcal{F}_A$  is called the *spectral transform* (or also a spectral representation of  $A$ ).

*Remarks.* 1. It is always possible to choose a fundamental system of solutions  $\Phi(\lambda, \cdot)$  that depends continuously (even analytically) on  $\lambda$  [42]. The spectral measure can then be constructed explicitly from the knowledge of such a set of fundamental solutions  $(A - z)\phi = 0$ , see [9, 39, 41, 42]. We will explain some details of this construction in Appendix A.

2. It can be shown that under mild conditions the measure  $d\rho$  is absolutely continuous with respect to Lebesgue measure, see [36, Thm. 3.2].

### 3. BASIC PROPERTIES OF PALEY-WIENER FUNCTIONS

In this section we define the Paley-Wiener spaces and describe some of their elementary properties.

**Definition 3.1.** Assume that  $\Lambda \subseteq \mathbb{R}$  with finite measure and that  $A$  is a self-adjoint realization of a differential expression of Sturm Liouville type. A function  $f \in L^2(\mathbb{R})$  is in the Paley-Wiener space  $PW_{\Lambda}(A)$  (or  $\Lambda$ -bandlimited with respect to  $A$ ), if

$$(3.1) \quad f = c_{\Lambda}(A)f.$$

Definition 3.1 is a special case of Pesenson's abstract notion of bandwidth [32] for general self-adjoint operators on a Hilbert space. Subsequently Paley-Wiener spaces became an important notion in many investigations in analysis, see [15, 34, 45, 23] for some examples. Our main contribution is the detailed investigation of the Paley-Wiener space associated to the Sturm-Liouville operator  $A_p = -(pf')'$  and their interpretation as spaces of variable bandwidth. Our main interest is the subtle dependence of these spaces on the bandwidth parametrization  $p$  and corresponding sampling results.

Using the spectral theory of Sturm-Liouville operators, the Paley-Wiener spaces  $PW_{\Lambda}(A)$  possess characterizations similar to the standard spaces of bandlimited functions.

**Proposition 3.2.** Assume that  $\Lambda \subseteq \mathbb{R}_0^+$  with finite measure and that  $A \geq 0$  is a self-adjoint realization of a differential expression of Sturm Liouville type with spectral measure  $\rho$  and corresponding spectral transform  $\mathcal{F}_A$ . Then the following are equivalent:

- (i)  $f \in PW_{\Lambda}(A)$ ,
- (ii)  $\text{supp } \mathcal{F}_A f \subseteq \Lambda$ ,
- (iii) there exists a function  $F \in L^2(\Lambda, d\rho)$  such that

$$(3.2) \quad f(x) = \int_{\Lambda} F(\lambda) \cdot \Phi(\lambda, x) d\rho(\lambda) \quad \text{a.e. } x \in \mathbb{R}.$$

If the spectral set is an interval,  $\Lambda = [0, \Omega]$ , then also the following conditions are equivalent to (i) — (iii):

(iv) Bernstein's inequality:  $\|A^k f\| \leq \Omega^k \|f\|$  for all  $k \in \mathbb{N}$ .

(v) Analyticity: For all  $g \in L^2(\mathbb{R})$  the function  $z \in \mathbb{C} \rightarrow \langle e^{zA} f, g \rangle$  is an entire function of exponential type  $\Omega$ , i.e., for all  $\epsilon > 0$

$$|\langle e^{zA} f, g \rangle| = \mathcal{O}(e^{(\Omega+\epsilon)|\Im z|}).$$

*Proof.* (i)  $\Rightarrow$  (ii), (iii): Since  $f = c_\Lambda(A)f$  for  $f \in PW_\Lambda(A)$ , (2.8) implies that

$$f(x) = \int_{\mathbb{R}_0^+} \mathcal{F}_A f(\lambda) \cdot \Phi(\lambda, x) d\rho(\lambda) = c_\Lambda(A)f(x) = \int_{\mathbb{R}_0^+} c_\Lambda(\lambda) \mathcal{F}_A f(\lambda) \cdot \Phi(\lambda, x) d\rho(\lambda).$$

Since  $\mathcal{F}_A^{-1}$  is unitary and thus one-to-one, this identity implies that  $\mathcal{F}_A f(\lambda) = c_\Lambda(\lambda) \mathcal{F}_A f(\lambda)$  for  $\rho$ -almost all  $\lambda$ , whence  $\text{supp } \mathcal{F}_A f \subseteq \Lambda$ .

(iii)  $\Rightarrow$  (i) Conversely, if  $f$  is represented by (3.2), then  $c_\Lambda(A)f = f$  and thus  $f \in PW_\Lambda(A)$ .

The equivalence (i)  $\Leftrightarrow$  (iv) follows directly from the spectral theorem for self-adjoint unbounded operators and was first proved in [32] for an abstract notion of bandlimitedness. The characterization (i)  $\Leftrightarrow$  (v) is proved in [34, 45]. See also [15] for a related characterization of bandwidth.  $\square$

**Proposition 3.3.** *Let  $\Lambda$  be a subset of  $\mathbb{R}_0^+$  with finite measure, and  $A \geq 0$  a self-adjoint realization of a differential expression of Sturm Liouville type.*

(i) *Then the Paley-Wiener space  $PW_\Lambda(A)$  is a closed subspace of  $L^2(\mathbb{R})$ . Every function in  $PW_\Lambda(A)$  is continuous.*

(ii) *If  $\Lambda$  is compact, the Paley-Wiener space  $PW_\Lambda(A)$  is a reproducing kernel Hilbert space; its kernel is*

$$(3.3) \quad k_\Lambda(x, y) = k(x, y) = \int_\Lambda \overline{\Phi(\lambda, x)} \cdot \Phi(\lambda, y) d\rho(\lambda),$$

*and  $k$  is the integral kernel of the spectral projection  $c_\Lambda(A)$  from  $L^2(\mathbb{R})$  onto  $PW_\Lambda(A)$ . The kernel  $k$  is continuous in  $x$  and  $y$ .*

*Proof.* We apply the Cauchy-Schwarz inequality for  $L^2(\mathbb{R}, d\rho)$  [9, XIII.5.8] to (3.2) and obtain

$$(3.4) \quad |f(x)| \leq \|F\|_{L^2(\Lambda, d\rho)} \|\Phi(\cdot, x)\|_{L^2(\Lambda, d\rho)}.$$

Thus the pointwise evaluation  $f \mapsto f(x)$  is continuous on  $PW_\Lambda(A)$  and  $PW_\Lambda(A)$  is a reproducing kernel Hilbert space. Since  $\Phi(\lambda, x)$  is continuous in  $\lambda$ , the continuity of  $f$  follows from classical facts about parameter integrals.

The formula (3.3) for the reproducing kernel is proved in [9, XIII.5.24]. It follows directly from the identity

$$\begin{aligned} P_\Lambda(A)f(x) &= \int_\Lambda \mathcal{F}_A f(\lambda) \cdot \overline{\Phi(\lambda, x)} d\rho(\lambda) \\ &= \int_\Lambda \int_{\mathbb{R}} f(y) \Phi(\lambda, y) \cdot \overline{\Phi(\lambda, x)} d\rho(\lambda) dy \\ &= \int_{\mathbb{R}} f(y) \overline{k_\Omega(x, y)} dy, \end{aligned}$$

after justifying the interchange of the integrals.  $\square$

*Remark.* The compactness condition in (3.3) above can be relaxed in various important cases. The inequality (3.4) holds if  $\|\Phi(\cdot, x)\|_{L^2(\Lambda, d\rho)}$  is finite. This is the case under the following set of conditions: (i)  $|\Lambda| < \infty$ , (ii) the spectral measure is absolutely continuous with respect to Lebesgue measure, and (iii) the solutions  $\Phi(\cdot, x)$  are bounded for every  $x$ . In particular this is true for the Schrödinger operator (2.5) with compactly supported potential  $q$ . We will use this fact in Section 6 (Proposition 6.8 and Equation (6.14)).

As mentioned in the introduction, a function in  $PW_{[0, \Omega]}(A_p)$  behaves locally like a function with bandwidth  $(\Omega/p(x))^{1/2}$ . The next result provides a precise formulation for this vague idea. Before its statement we recall that a function  $f$  belongs to the Bernstein space  $B_\Omega$ , if its distributional Fourier transforms has support in  $[-\Omega, \Omega]$ . By the Paley-Wiener theorem for distributions (see, e.g., [35])  $f \in B_\Omega$ , if and only if  $f$  can be extended from  $\mathbb{R}$  to an entire function  $F$  of exponential type  $\Omega$ , i.e.,  $F|_{\mathbb{R}} = f$  and  $|F(z)| = \mathcal{O}(e^{(\Omega+\epsilon)|\operatorname{Im} z|})$ .

**Proposition 3.4.** *If  $p(x) = p_0$  for  $x$  in an open interval  $I$ , then on  $I$  every  $f \in PW_{[0, \Omega]}(A_p)$  coincides with a function in  $B_{\sqrt{\Omega/p_0}}$  restricted to  $I$ .*

*Proof.* Since  $p|_I = p_0$ , the restriction of the differential expression  $\tau_p$  to  $I$  is just  $-p_0 \frac{d^2}{dx^2}$  and therefore the eigenvalue equation  $(\tau_p - \lambda)\phi = 0$  possesses the solutions  $e^{\pm i\sqrt{\lambda/p_0}x}$  on  $I$ . Let  $\tilde{\Phi}(\lambda, x) = (e^{i\sqrt{\lambda/p_0}x}, e^{-i\sqrt{\lambda/p_0}x})$  for all  $x \in \mathbb{R}$ , i.e.,  $\tilde{\Phi}(\lambda, \cdot)$  is a fundamental system of  $(-p_0 \frac{d^2}{dx^2} - \lambda)\phi = 0$  on  $\mathbb{R}$ . We may now choose a fundamental system  $\Phi(\lambda, \cdot)$  of  $(\tau_p - \lambda)\phi = 0$  and a corresponding spectral measure  $\rho$ , such that  $\Phi(\lambda, x) = \tilde{\Phi}(\lambda, x)$  for  $x \in I$ . By (3.2) every  $f \in PW_\Lambda(A_p)$  has a representation  $f(x) = \int_0^\Omega F(\lambda) \cdot \Phi(\lambda, x) d\rho(\lambda)$  for some  $F \in L^2([0, \Omega], d\rho)$ . Now set

$$\tilde{f}(x) = \int_0^\Omega F(\lambda) \cdot \tilde{\Phi}(\lambda, x) d\rho(\lambda).$$

Then  $f(x) = \tilde{f}(x)$  for  $x \in I$ .

By definition,  $\tilde{\Phi}(\lambda, \cdot)$  can be extended to an entire function that satisfies  $|\tilde{\Phi}(\lambda, z)| \leq 2e^{\lambda|\Im z|}$ . Therefore the vector-valued Hölder inequality implies that  $\tilde{f}$  also extends to a function on  $\mathbb{C}$  obeying the growth estimate

$$(3.5) \quad |\tilde{f}(z)| \leq \|F\|_{L^1([0, \Omega], d\rho)} \|\Phi(\cdot, z)\|_{L^\infty([0, \Omega], d\rho)} \leq Ce^{\sqrt{\frac{\Omega}{p_0}}|\Im z|}$$

for  $z \in \mathbb{C}$ . Clearly, the restriction of  $\tilde{f}$  to  $\mathbb{R}$  is bounded, and the parameter integral  $z \mapsto \tilde{f}(z)$  is analytic in  $z$  (use Morera's theorem). Therefore  $\tilde{f}$  belongs to the Bernstein space  $B_{\sqrt{\Omega/p_0}}$  and  $\tilde{f}|_I = f|_I$ , as claimed.  $\square$

#### 4. A TOY EXAMPLE

In this section we assume that  $\Lambda = [0, \Omega] \subseteq \mathbb{R}^+$  and study the special case

$$p(x) = \begin{cases} p_-, & x \leq 0 \\ p_+, & x > 0. \end{cases}$$

Our point is that all formulas are explicit and thus may help build the reader's intuition for variable bandwidth.

Using the continuity of solutions  $\phi$  of the differential equation  $(\tau_p - z)\phi = 0$  and the continuity of  $p(x)\phi'(x)$  at  $x = 0$ , we obtain the linearly independent solutions

$$\begin{aligned} \phi_+(z, x) &= \begin{cases} \frac{1}{2} \left( 1 + \sqrt{\frac{p_+}{p_-}} \right) e^{i\sqrt{z/p_-}x} + \frac{1}{2} \left( 1 - \sqrt{\frac{p_+}{p_-}} \right) e^{-i\sqrt{z/p_-}x}, & x \leq 0, \\ e^{i\sqrt{z/p_+}x}, & x > 0, \end{cases} \\ \phi_-(z, x) &= \begin{cases} e^{-i\sqrt{z/p_-}x}, & x \leq 0, \\ \frac{1}{2} \left( 1 + \sqrt{\frac{p_-}{p_+}} \right) e^{-i\sqrt{z/p_+}x} + \frac{1}{2} \left( 1 - \sqrt{\frac{p_-}{p_+}} \right) e^{i\sqrt{z/p_+}x}, & x > 0, \end{cases} \end{aligned}$$

by a straightforward calculation. For  $\Im z > 0$  the solution  $\phi_+$  lies right in  $L^2(\mathbb{R})$  and  $\phi_-$  lies left in  $L^2(\mathbb{R})$ . Note that for real  $z = \lambda \in \mathbb{R}$  and  $x \leq 0$  the solution  $\phi_+$  can be written as

$$\phi_+(\lambda, x) = \cos\left(\sqrt{\frac{\lambda}{p_-}}x\right) + i\sqrt{\frac{p_+}{p_-}} \sin\left(\sqrt{\frac{\lambda}{p_-}}x\right),$$

and similarly for  $\phi_-$  for  $x \geq 0$ .

We can derive the spectral measure explicitly to be

$$(4.1) \quad d\rho(\lambda) = \frac{1}{\pi(\sqrt{p_-} + \sqrt{p_+})^2} \begin{pmatrix} \sqrt{p_-} & 0 \\ 0 & \sqrt{p_+} \end{pmatrix} \frac{d\lambda}{\sqrt{\lambda}}.$$

The details of this computation are sketched in Appendix A.

Using (3.3) and  $\text{sinc}(x) = \sin x/x$ , a straightforward computation leads to the reproducing kernel

$$(4.2) \quad k(x, y) = \begin{cases} \frac{\Omega^{1/2}}{\pi\sqrt{p_-}} \text{sinc}\left(\Omega^{1/2} \frac{x-y}{\sqrt{p_-}}\right) - \frac{\sqrt{p_+}-\sqrt{p_-}}{\sqrt{p_+}+\sqrt{p_-}} \frac{\Omega^{1/2}}{\pi\sqrt{p_-}} \text{sinc}\left(\Omega^{1/2} \frac{x+y}{\sqrt{p_-}}\right) & x, y \leq 0, \\ \frac{\Omega^{1/2}}{\pi\sqrt{p_+}} \text{sinc}\left(\Omega^{1/2} \frac{x-y}{\sqrt{p_+}}\right) + \frac{\sqrt{p_+}-\sqrt{p_-}}{\sqrt{p_+}+\sqrt{p_-}} \frac{\Omega^{1/2}}{\pi\sqrt{p_+}} \text{sinc}\left(\Omega^{1/2} \frac{x+y}{\sqrt{p_+}}\right) & x, y > 0, \\ \frac{2\Omega^{1/2}}{\pi(\sqrt{p_+}+\sqrt{p_-})} \text{sinc}\left(\Omega^{1/2}\left(\frac{x}{\sqrt{p_-}} - \frac{y}{\sqrt{p_+}}\right)\right) & x \leq 0, y > 0, \\ \frac{2\Omega^{1/2}}{\pi(\sqrt{p_+}-\sqrt{p_-})} \text{sinc}\left(\Omega^{1/2}\left(\frac{x}{\sqrt{p_+}} - \frac{y}{\sqrt{p_-}}\right)\right) & x > 0, y \leq 0. \end{cases}$$

If  $p_- = p_+ = 1$  we obtain the sinc kernel, as expected.

Direct inspection of (4.2) suggests a recipe to obtain an orthonormal basis of  $PW_{[0, \Omega]}(A_p)$ . As a result we obtain a sampling formula that is similar to the cardinal series for bandlimited functions.

**Theorem 4.1.** Fix the sampling set  $\{x_j = \frac{\pi j \sqrt{p(j)}}{\Omega^{1/2}} : j \in \mathbb{Z}\}$  and the weights

$$(4.3) \quad w_j = \begin{cases} \sqrt{p_-}, & j < 0, \\ \sqrt{p_+}, & j > 0, \\ \frac{1}{2}(\sqrt{p_+} + \sqrt{p_-}), & j = 0. \end{cases}$$

Then the set  $\{\sqrt{\frac{\pi w_j}{\Omega^{1/2}}} k(x_j, \cdot) : j \in \mathbb{Z}\}$  forms an orthonormal basis of  $PW_{[0, \Omega]}(A_p)$ . The orthogonal expansion

$$(4.4) \quad f(x) = \frac{\pi}{\Omega^{1/2}} \sum_{j \in \mathbb{Z}} w_j f(x_j) k(x_j, x)$$

converges in  $L^2(\mathbb{R})$  and uniformly for every  $f \in PW_{[0, \Omega]}(A_p)$ .

*Proof.* The orthogonality follows directly from  $\langle k(x_i, \cdot), k(x_j, \cdot) \rangle = k(x_i, x_j)$  and the formulas for the reproducing kernel (4.2). For  $i = j$  we obtain  $\langle k(x_j, \cdot), k(x_j, \cdot) \rangle = k(x_j, x_j) = \frac{\Omega^{1/2}}{\pi w_j} = \frac{\Omega^{1/2}}{\pi \sqrt{p_{\pm}}}$ , and thus  $\{\sqrt{\frac{\pi w_j}{\Omega^{1/2}}} k(x_j, \cdot) : j \in \mathbb{Z}\}$  is an orthonormal set.

To prove the completeness of the orthogonal system, assume that  $f \in PW_{\Lambda}(A_p)$  and  $\langle f, k(x_j, \cdot) \rangle = 0$  for all  $j \in \mathbb{Z}$ . We have to show that  $f \equiv 0$ .

Using the unitarity of  $\mathcal{F}_{A_p}$ , this is equivalent to proving that  $F = (F_1, F_2) \in L^2(\Lambda, d\rho)$  and  $\langle F, \Phi(\cdot, x_j) \rangle_{L^2(\Lambda, d\rho)} = 0$  for all  $j \in \mathbb{Z}$  implies  $F \equiv 0$ .

We substitute the fundamental solutions in the inner product and make the change of variables  $\lambda = \omega^2$ ,  $d\lambda/\sqrt{\lambda} = 2d\omega$ ; then the vanishing of  $\langle F, \Phi(\cdot, x_j) \rangle = 0$  amounts to the following conditions:

$$\begin{aligned} \int_0^{\Omega^{1/2}} (\sqrt{p_-} F_1(\omega^2) + \sqrt{p_+} F_2(\omega^2)) \cos \frac{\pi j \omega}{\Omega^{1/2}} d\omega + i \sqrt{p_+} \int_0^{\Omega^{1/2}} (F_1(\omega^2) - F_2(\omega^2)) \sin \frac{\pi j \omega}{\Omega^{1/2}} d\omega &= 0, \quad j < 0, \\ \int_0^{\Omega^{1/2}} (\sqrt{p_-} F_1(\omega^2) + \sqrt{p_+} F_2(\omega^2)) \cos \frac{\pi j \omega}{\Omega^{1/2}} d\omega + i \sqrt{p_-} \int_0^{\Omega^{1/2}} (F_1(\omega^2) - F_2(\omega^2)) \sin \frac{\pi j \omega}{\Omega^{1/2}} d\omega &= 0, \quad j \geq 0, \end{aligned}$$

If we re-index the first equation ( $j \rightarrow -j$ ) we obtain

$$\int_0^{\Omega^{1/2}} (\sqrt{p_-} F_1(\omega^2) + \sqrt{p_+} F_2(\omega^2)) \cos \frac{\pi j \omega}{\Omega^{1/2}} d\omega - i \sqrt{p_+} \int_0^{\Omega^{1/2}} (F_1(\omega^2) - F_2(\omega^2)) \sin \frac{\pi j \omega}{\Omega^{1/2}} d\omega = 0, \quad j > 0.$$

Adding and subtracting the above equations yields

$$(4.5) \quad \int_0^{\Omega^{1/2}} (\sqrt{p_-} F_1(\omega^2) + \sqrt{p_+} F_2(\omega^2)) \cos \frac{\pi j \omega}{\Omega^{1/2}} d\omega = 0, \quad j \geq 0,$$

$$(4.6) \quad \int_0^{\Omega^{1/2}} (F_1(\omega^2) - F_2(\omega^2)) \sin \frac{\pi j \omega}{\Omega^{1/2}} d\omega = 0, \quad j > 0.$$

Equation (4.5) describes the Fourier cosine coefficients of the even continuation of the function  $\omega \mapsto \sqrt{p_-} F_1(\omega^2) + \sqrt{p_+} F_2(\omega^2)$  to  $[-\Omega^{1/2}, \Omega^{1/2}]$ . Consequently, from the uniqueness of Fourier cosine series we obtain  $\sqrt{p_-} F_1(\omega^2) + \sqrt{p_+} F_2(\omega^2) = 0$  on  $[0, \Omega^{1/2}]$ . Likewise, (4.6) with an odd extension of the integrand, the uniqueness

theorem for Fourier sine series yields  $F_1(\omega^2) - F_2(\omega^2) = 0$  on  $[0, \Omega^{1/2}]$ . Combining these two results and substituting back we obtain  $F_1 = F_2 = 0$  on  $[0, \Omega]$ .

We have proved that the set  $\{\sqrt{\frac{\pi w_j}{\Omega^{1/2}}} k(x_j, \cdot) : j \in \mathbb{Z}\}$  is an orthonormal basis of  $PW_\Lambda(A_p)$ . Therefore every  $f \in PW_\Lambda(A_p)$  possesses the orthonormal expansion

$$f = \sum_{j \in \mathbb{Z}} \frac{\pi w_j}{\Omega^{1/2}} \langle f, k(x_j, \cdot) \rangle k(x_j, \cdot) = \frac{\pi}{\Omega^{1/2}} \sum_{j \in \mathbb{Z}} w_j f(x_j) k(x_j, \cdot)$$

with convergence in norm. In a reproducing kernel Hilbert space, the  $L^2$ -convergence implies pointwise convergence. Since the family  $\{k(x_j, \cdot) : j \in \mathbb{Z}\}$  is norm-bounded, it also implies uniform convergence (see [29, 3.1]).  $\square$

*Remark.* It would be of great interest to construct an orthonormal basis of reproducing kernels and a corresponding similar sampling theorem for more general control functions  $p$ .

Let us now consider the case  $p_- \rightarrow \infty$ ,  $p_+ = 1$ . Proceeding formally from (4.2), the reproducing kernel of  $PW_{[0, \Omega]}(A_p)$  is

$$k(x, y) = \begin{cases} \frac{\Omega^{1/2}}{\pi} [\text{sinc}(\Omega^{1/2}(x - y)) - \text{sinc}(\Omega^{1/2}(x + y))], & x, y \geq 0, \\ 0, & \text{else.} \end{cases}$$

We see that every  $f \in PW_{[0, \Omega]}(A_p)$  has its support in  $[0, \infty)$ . For a more concrete interpretation of  $PW_{[0, \Omega]}(A_p)$  let  $g$  be the extension of  $f$  to an odd function on  $\mathbb{R}$ . Then  $f \in PW_{[0, \Omega]}(A_p)$  holds, if and only if

$$f(x) = \int_0^\infty f(y) k(x, y) dy = \int_{\mathbb{R}} g(y) \text{sinc}(\Omega^{1/2}(x - y)) dy \quad x \geq 0.$$

This means that  $f$  can be interpreted as the restriction of an odd bandlimited function  $g$  with  $\text{supp } \hat{g} \subseteq [-\Omega^{1/2}, \Omega^{1/2}]$  to  $[0, \infty)$ .

Theorem 4.1 then yields the following sampling theorem.

**Proposition 4.2.** *If  $p(x) = \infty$  for  $x \leq 0$  and  $p(x) = 1$  for  $x > 0$ , the following sampling formula holds for  $f \in PW_{[0, \Omega]}(A_p)$*

$$f(x) = \frac{\pi}{\Omega^{1/2}} \sum_{j=1}^{\infty} f\left(\frac{\pi j}{\Omega^{1/2}}\right) k\left(\frac{\pi j}{\Omega^{1/2}}, x\right) = \sum_{j=1}^{\infty} (-1)^j f\left(\frac{\pi j}{\Omega^{1/2}}\right) \sin(\Omega^{1/2} x) \frac{2\pi j}{(\Omega^{1/2} x)^2 - (\pi j)^2}.$$

We do not give a formal proof for the limiting procedure. The corollary follows directly from the observation that  $f \in PW_{[0, \Omega]}(A_p)$ , if and only if  $f$  is the restriction of an odd function  $g$  with  $\text{supp } \hat{g} \subseteq [-\Omega^{1/2}, \Omega^{1/2}]$  to  $\mathbb{R}^+$ . Thus

$$\begin{aligned} f(x) &= \frac{\pi}{\Omega^{1/2}} \sum_{j \in \mathbb{Z}} f\left(\frac{\pi j}{\Omega^{1/2}}\right) k\left(\frac{\pi j}{\Omega^{1/2}}, x\right) \\ &= \frac{\pi}{\Omega^{1/2}} \sum_{j=1}^{\infty} f\left(\frac{\pi j}{\Omega^{1/2}}\right) \left( \text{sinc}\left(\Omega^{1/2}\left(x - \frac{\pi j}{\Omega^{1/2}}\right)\right) - \text{sinc}\left(\Omega^{1/2}\left(x + \frac{\pi j}{\Omega^{1/2}}\right)\right) \right). \end{aligned}$$

## 5. NONUNIFORM SAMPLING

In this section we assume that  $\Lambda \subseteq [0, \Omega] \subset \mathbb{R}^+$ . Based on the method developed in [12, 13], we derive a sampling theorem and reconstruction procedures for  $PW_\Lambda(A_p)$ . All constants and error estimates are explicit and highlight the role of the parametrizing function  $p$ .

Given a set  $X = \{x_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}$ , we denote by  $y_i = \frac{1}{2}(x_i + x_{i+1})$ ,  $i \in \mathbb{Z}$ , the midpoints of  $X$ , and set  $\chi_i = c_{[y_{i-1}, y_i]}$ . Then the functions  $\chi_i$  form a partition of unity. We also set  $I'_i = [y_{i-1}, x_i)$  and  $I''_i = [x_i, y_i)$ . Let  $\delta$  be the maximum gap between consecutive sampling points weighted by the parametrizing function  $p$ , namely

$$(5.1) \quad \delta = \sup_{i \in \mathbb{Z}} \frac{x_{i+1} - x_i}{\inf_{x \in [x_i, x_{i+1}]} \sqrt{p(x)}}.$$

We first derive a fundamental inequality for functions in  $PW_\Lambda(A_p)$ . For the proof we need Wirtinger's inequality, see, e.g., [19]. If  $f, f' \in L^2([a, b])$  and either  $f(a) = 0$  or  $f(b) = 0$ , then

$$(5.2) \quad \int_a^b |f(x)|^2 dx \leq \frac{4}{\pi^2} (b - a)^2 \int_a^b |f'(x)|^2 dx.$$

**Lemma 5.1.** *Let  $\Lambda \subseteq [0, \Omega] \subset \mathbb{R}^+$  and assume that  $\inf_{x \in \mathbb{R}} p(x) > 0$ . If  $\delta$  is finite, then for all  $f \in PW_\Lambda(A_p)$  we have*

$$(5.3) \quad \left\| f - \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \right\|^2 \leq \frac{\delta^2 \Omega}{\pi^2} \|f\|^2.$$

Consequently,  $PW_\Lambda(A_p)$  satisfies a Plancherel-Polya inequality of the form

$$(5.4) \quad \sum_{i \in \mathbb{Z}} |f(x_i)|^2 \frac{x_{i+1} - x_{i-1}}{2} \leq \left(1 + \frac{\delta \Omega^{1/2}}{\pi}\right)^2 \|f\|^2, \quad f \in PW_\Lambda(A_p).$$

*Proof.* The proof is an adaption of the proof of [12, Thm. 1]. We rewrite the expression (5.3) as follows:

$$(5.5) \quad \begin{aligned} \left\| f - \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \right\|^2 &= \left\| \sum_{i \in \mathbb{Z}} (f - f(x_i)) \chi_i \right\|^2 = \sum_{i \in \mathbb{Z}} \int_{y_{i-1}}^{y_i} |f(x) - f(x_i)|^2 dx \\ &= \sum_{i \in \mathbb{Z}} \left( \int_{y_{i-1}}^{x_i} |f(x) - f(x_i)|^2 dx + \int_{x_i}^{y_i} |f(x) - f(x_i)|^2 dx \right). \end{aligned}$$



By Wirtinger's inequality (5.2) this can be estimated further as

$$\begin{aligned}
\left\| f - \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \right\|^2 &\leq \frac{4}{\pi^2} \sum_{i \in \mathbb{Z}} \left[ (x_i - y_{i-1})^2 \int_{y_{i-1}}^{x_i} |f'(x)|^2 dx + (y_i - x_i)^2 \int_{x_i}^{y_i} |f'(x)|^2 dx \right] \\
&\leq \frac{4}{\pi^2} \sum_{i \in \mathbb{Z}} \left[ \frac{(x_i - y_{i-1})^2}{\min_{x \in I'_i} p(x)} \int_{y_{i-1}}^{x_i} p(x) |f'(x)|^2 dx + \frac{(y_i - x_i)^2}{\min_{x \in I''_i} p(x)} \int_{x_i}^{y_i} p(x) |f'(x)|^2 dx \right] \\
&\leq \frac{4}{\pi^2} \sup_{i \in \mathbb{Z}} \max \left( \frac{(x_i - y_{i-1})^2}{\min_{x \in I'_i} p(x)}, \frac{(y_i - x_i)^2}{\min_{x \in I''_i} p(x)} \right) \sum_{i \in \mathbb{Z}} \int_{y_{i-1}}^{y_i} p(x) |f'(x)|^2 dx \\
&= \frac{4}{\pi^2} \sup_{i \in \mathbb{Z}} \left( \frac{(x_i - y_{i-1})^2}{\min_{x \in I'_i} p(x)}, \frac{(y_i - x_i)^2}{\min_{x \in I''_i} p(x)} \right) \int_{\mathbb{R}} p(x) |f'(x)|^2 dx \\
&\leq \frac{1}{\pi^2} \sup_{i \in \mathbb{Z}} \max \frac{(y_i - y_{i-1})^2}{\min_{x \in [x_{i-1}, x_i]} p(x)} \int_{\mathbb{R}} p(x) |f'(x)|^2 dx.
\end{aligned}$$

As  $f \in PW_\Lambda(A_p) \subseteq \mathcal{D}(A_p)$  we can simplify the last term by using integration by parts and then apply Bernstein's inequality (Proposition 3.2 (iv)).

$$(5.6) \quad \int_{\mathbb{R}} p(x) |f'(x)|^2 dx = \langle A_p f, f \rangle \leq \|A_p f\| \|f\| \leq \Omega \|f\|^2.$$

The decisive modification was to smuggle in the parametrizing function  $p$  to obtain  $\int p |f'|^2$  and then to apply Bernstein's inequality.

In conclusion we obtain

$$(5.7) \quad \left\| f - \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \right\|^2 \leq \frac{1}{\pi^2} \sup_{i \in \mathbb{Z}} \frac{(x_i - x_{i-1})^2}{\min_{x \in [x_{i-1}, x_i]} p(x)} \Omega \|f\|^2 = \frac{\delta^2 \Omega}{\pi^2} \|f\|^2$$

for  $f \in PW_\Lambda(A_p)$ , and (5.4) follows.  $\square$

The fundamental inequality implies immediately a sampling theorem for  $PW_\Lambda(A_p)$ .

**Theorem 5.2** (Sampling inequality). *Let  $\Lambda \subseteq [0, \Omega] \subseteq \mathbb{R}_0^+$  and assume that  $\inf_{x \in \mathbb{R}} p(x) > 0$ . If*

$$(5.8) \quad \delta = \sup_{i \in \mathbb{Z}} \frac{x_{i+1} - x_i}{\inf_{x \in [x_i, x_{i+1}]} \sqrt{p(x)}} < \frac{\pi}{\Omega^{1/2}},$$

then, for all  $f \in PW_\Lambda(A_p)$ ,

$$(5.9) \quad \left(1 - \frac{\delta \Omega^{1/2}}{\pi}\right)^2 \|f\|^2 \leq \sum_{i \in \mathbb{Z}} \frac{x_{i+1} - x_{i-1}}{2} |f(x_i)|^2 \leq \left(1 + \frac{\delta \Omega^{1/2}}{\pi}\right)^2 \|f\|^2.$$

If, in addition,  $\inf_{i \in \mathbb{Z}} (x_{i+1} - x_i) = \gamma > 0$  ( $X$  is separated), then  $X$  is a set of stable sampling for  $PW_\Lambda(A_p)$  with lower bound  $\gamma(1 - \frac{\delta \Omega^{1/2}}{\pi})^2$ . Equivalently, the set of reproducing kernels  $\{k(x, \cdot) : x \in X\}$  is a frame for  $PW_\Lambda(A_p)$ .

*Proof.* We use the triangle inequality and (5.3) to obtain

$$\begin{aligned} \sum_{i \in \mathbb{Z}} |f(x_i)|^2 \frac{x_{i+1} - x_{i-1}}{2} &= \left\| \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \right\|^2 \\ &\geq \left( \|f\| - \left\| f - \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \right\| \right)^2 \geq \left( 1 - \frac{\delta \Omega^{1/2}}{\pi} \right)^2 \|f\|^2. \end{aligned}$$

The upper bound is already in (5.4).  $\square$

Based on the sampling inequality (5.9), one may formulate several algorithms for the reconstruction of  $f \in PW_\Lambda(A_p)$  from its samples  $f(x_i), i \in \mathbb{Z}$ . On the one hand one may use the manifold variations of the frame algorithm (iterative, accelerated iterations, or by means of a dual frame), on the other hand, one may use the following iterative algorithm from [12]. We set  $P_\Lambda = c_\Lambda(A_p)$  for the orthogonal projection onto  $PW_\Lambda(A_p)$ .

**Theorem 5.3.** *Let  $\Lambda \subseteq [0, \Omega] \subseteq \mathbb{R}^+$  and assume that  $\inf_{x \in \mathbb{R}} p(x) > 0$ . Assume that the sampling set  $X$  satisfies the maximum gap condition (5.8). Then  $f \in PW_\Lambda(A_p)$  can be reconstructed from its sampled values  $(f(x_i))_{i \in \mathbb{Z}}$  by the following algorithm.*

*Initialization:*  $f_0 = h_0 = P_\Lambda \left( \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \right)$ .

*Iteration:*  $h_{n+1} = h_n - P_\Lambda \left( \sum_{i \in \mathbb{Z}} h_n(x_i) \chi_i \right)$  for  $n \geq 0$ .

*Update:*  $f_{n+1} = \sum_{j=0}^{n+1} h_j = f_n + h_{n+1}$  for  $n \geq 0$ .

*Then*

$$(5.10) \quad f = \lim_{n \rightarrow \infty} f_n = \sum_{n=0}^{\infty} h_n,$$

*with the error estimate*

$$(5.11) \quad \|f - f_n\| \leq \left( \frac{\delta \Omega^{1/2}}{\pi} \right)^{n+1} \frac{\pi + \delta \Omega^{1/2}}{\pi - \delta \Omega^{1/2}} \|f\|.$$

*Proof.* Define  $R$  by  $Rf = P_\Lambda \left( \sum_{i \in \mathbb{Z}} f(x_i) \chi_i \right)$ . By the Plancherel-Polya inequality (5.4) the operator is bounded on  $PW_\Lambda(A_p)$ . With this notation we have  $h_0 = Rf$  and the iteration step is

$$h_{n+1} = (I - R)h_n = (I - R)^{n+1}h_0 = (I - R)^{n+1}Rf.$$

The sum  $\sum h_n = \sum_{n=0}^{\infty} (I - R)^n Rf$  is just the Neumann series for the inverse of  $R$  applied to  $Rf$ . By the fundamental inequality (5.3) and the assumption on  $\delta$  the operator norm of  $I - R$  on  $PW_\Lambda(A_p)$  is bounded by  $\|I - R\| \leq \delta \Omega^{1/2}/\pi$ . Since  $\delta \Omega^{1/2}/\pi < 1$  by assumption, the Neumann series converges in the operator norm to  $R^{-1}$  and consequently  $f = \sum_{n=0}^{\infty} h_n$ . The error estimate follows from the properties of geometric series.  $\square$

*Remarks.* (1) If we replace the indicator functions in the reconstruction algorithm by partitions of unity with higher regularity with respect to  $A_p$  then the convergence rate of the approximations can be increased. The results require an adapted form

of Wirtinger's inequality and will be published elsewhere.

(2) The theorem requires only very weak regularity conditions for  $p$ , essentially  $p$  should be bounded away from zero.

## 6. LANDAU'S NECESSARY DENSITY CONDITIONS

In this section we state and prove density conditions in the style of Landau for sampling sequences  $X \subset \mathbb{R}$ , spectral sets  $\Lambda \subseteq \mathbb{R}^+$  of finite Lebesgue measure, and functions in  $PW_\Lambda(A_p)$ . We find necessary conditions on  $X$  in terms of an appropriately defined version of the Beurling density such that the reproducing kernels  $\{k(x, \cdot) : x \in X\}$  form either a frame or a Riesz sequence for  $PW_\Lambda(A_p)$ . Recall that  $\{k(x, \cdot) : x \in X\}$  is a Riesz sequence for  $PW_\Lambda(A_p)$ , if there are positive constants  $C, D$  such that for all  $c \in \ell^2(X)$

$$C \sum_{x \in X} |c_x|^2 \leq \left\| \sum_{x \in X} c_x k(x, \cdot) \right\|^2 \leq D \sum_{x \in X} |c_x|^2.$$

Equivalently, for all  $c \in \ell^2(X)$  there exists an  $f \in PW_\Lambda(A_p)$ , such that  $f(x) = c_x, \forall x \in X$ , therefore  $X$  is also called a *set of interpolation* for  $PW_\Lambda(A_p)$ .

**6.1. Beurling density.** Assume  $X$  is a *relatively separated* subset of  $\mathbb{R}$ , i.e.,  $\max_{c \in \mathbb{R}} \#(X \cap [c, c+1]) = n_0 < \infty$ . This property implies that an interval  $I$  of length  $|I|$  contains at most  $(|I| + 1)n_0$  points of  $X$ . The upper Beurling density of  $X$  is defined as

$$D^+(X) = \overline{\lim}_{r \rightarrow \infty} \sup_{|I|=r} \frac{\#\{X \cap I : I \subset \mathbb{R} \text{ closed interval}\}}{r},$$

and the lower Beurling density is

$$D^-(X) = \underline{\lim}_{r \rightarrow \infty} \inf_{|I|=r} \frac{\#\{X \cap I : I \subset \mathbb{R} \text{ closed interval}\}}{r}.$$

Landau's density conditions for the classical Paley-Wiener space  $PW_\Lambda = \{f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subseteq \Lambda\}$  state that a set of stable sampling  $X$  for  $PW_\Lambda$  satisfies necessarily  $D^-(X) \geq |\Lambda|/(2\pi)$ . Similarly, if  $X$  is a set of interpolation for  $PW_\Lambda$  then  $D^+(X) \leq |\Lambda|/(2\pi)$  [25, 26].

What is the appropriate notion of density for the Paley-Wiener spaces of variable bandwidth? Let us assume first that  $p$  is piecewise constant, say  $p(x) = p_k$  on the interval  $I_k$ . According to Proposition 3.4 the function  $f \in PW_{[0, \Omega]}(A_p)$  coincides on  $I_k$  with the restriction of a function in the Bernstein space  $B_{\sqrt{\Omega/p_k}}$  to  $I_k$ , so we expect the number of samples in  $I_k$  required for the reconstruction in  $I_k$  to be roughly

$$(6.1) \quad \frac{\#(X \cap I_k)}{|I_k|} \sim \sqrt{\frac{\Omega}{p_k}}.$$

Rewriting (6.1) as  $\frac{\#(X \cap I_k)}{|I_k| p_k^{-1/2}} \sim \Omega^{1/2}$ , we may interpret  $|I_k| p_k^{-1/2}$  as a new measure (or distance function) on  $\mathbb{R}$  and the quantity in (6.1) as an average number of samples with respect to this measure.

We therefore introduce the measure  $\mu_p$  by

$$(6.2) \quad \mu_p(I) = \int_I p^{-1/2}(u) du.$$

**Definition 6.1.** Assume that  $p^{-1/2} \in L^1_{loc}$  and that  $X \subseteq \mathbb{R}$  is  $\mu_p$ -separated, i.e.,  $\inf\{\mu_p([x, z]): x, z \in X, x < z\} > 0$ . The upper  $A_p$ -Beurling density is defined as

$$D_p^+(X) = \lim_{r \rightarrow \infty} \sup_{\mu_p(I)=r} \frac{\#\{X \cap I\} : I \subset \mathbb{R} \text{ closed interval}}{r},$$

and the lower  $A_p$ -Beurling density is

$$D_p^-(X) = \lim_{r \rightarrow \infty} \inf_{\mu_p(I)=r} \frac{\#\{X \cap I\} : I \subset \mathbb{R} \text{ closed interval}}{r}.$$

Again, for  $p \equiv 1$  these densities coincide with the standard Beurling densities.

To derive necessary density conditions for sampling and interpolation in  $PW_\Lambda(A_p)$ , we restrict our attention to the model case of eventually constant  $p$ . From now on we assume that  $p$  satisfies the universal assumption (2.2) and that

$$(6.3) \quad \begin{aligned} p, p' &\in AC_{loc}(\mathbb{R}), \\ p(x) &= \begin{cases} p_- > 0, & \text{if } x < -R \\ p_+ > 0, & \text{if } x > R. \end{cases} \end{aligned}$$

**Theorem 6.2** (Necessary density conditions for interpolation). *Assume that  $\Lambda \subseteq \mathbb{R}^+$  has finite Lebesgue measure and that  $p$  satisfies (6.3). If  $\{k(x, \cdot) : x \in X\}$  is a Riesz sequence for  $PW_\Lambda(A_p)$ , then*

$$D_p^+(X) \leq \frac{|\Lambda^{1/2}|}{\pi}.$$

**Theorem 6.3** (Necessary density conditions for sampling). *Assume that  $\Lambda \subseteq \mathbb{R}^+$  has finite Lebesgue measure and that  $p$  satisfies (6.3). If  $\{k(x, \cdot) : x \in X\}$  is a frame for  $PW_\Lambda(A_p)$ , then*

$$D_p^-(X) \geq \frac{|\Lambda^{1/2}|}{\pi}.$$

Thus the quantity  $\frac{|\Lambda^{1/2}|}{\pi}$  is the critical density that separates sets of stable sampling from sets of interpolation.

*Remarks.* 1. We have seen in the introduction that for  $p \equiv 1$ ,  $A_p = -\frac{d^2}{dx^2}$ , and  $\Lambda = [0, \Omega]$ , the corresponding Paley-Wiener space  $PW_{[0, \Omega]}(A_p)$  is equal to the space of bandlimited functions  $\{f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subseteq [-\Omega^{1/2}, \Omega^{1/2}]\}$ . Then the necessary density condition is  $D_p^-(X) = D^-(X) \geq \Omega^{1/2}/\pi = |[-\Omega^{1/2}, \Omega^{1/2}]|/(2\pi)$ . Theorem 6.3 contains Landau's result as a special case. The difference in formulation comes from the use of a second order differential operator which identifies positive and negative frequencies with a single spectral value.

2. The assumption (6.3) excludes both the toy example of Section 4 (because of lacking smoothness) and more general parametrizing functions (when  $p$  tends to

$p_{\pm}$  at a certain rate). To restrict the length of this paper, we will only treat the case of eventually constant  $p$  and return to weaker assumptions in our future work.

We first compare the maximum gap condition of Section 5 with the Beurling density.

**Proposition 6.4.** *Let  $X = \{x_i : i \in \mathbb{Z}\} \subseteq \mathbb{R}$  be a set with  $x_i < x_{i+1}$  for all  $i$  and  $\lim_{i \rightarrow \pm\infty} x_i = \pm\infty$ . If*

$$(6.4) \quad \sup_i \frac{x_{i+1} - x_i}{\inf_{x \in [x_i, x_{i+1}]} \sqrt{p(x)}} = \eta$$

then  $D_p^-(X) \geq \eta^{-1}$ .

*Proof.* The gap condition (6.4) implies that for all  $i \in \mathbb{Z}$

$$\int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{p(x)}} dx \leq \frac{x_{i+1} - x_i}{\inf_{x \in [x_i, x_{i+1}]} \sqrt{p(x)}} \leq \eta.$$

Given a bounded, closed interval  $I \subseteq \mathbb{R}$ , let  $i_0 = \min\{i \in \mathbb{Z} : x_i \in I\}$  and  $i_1 = \max\{i \in \mathbb{Z} : x_i \in I\}$  the smallest and largest indices of  $x_i \in I$ . Then  $I \subseteq [x_{i_0-1}, x_{i_1+1}]$  and

$$\mu_p(I) = \int_I \frac{1}{\sqrt{p(x)}} dx \leq \sum_{i=i_0-1}^{i_1} \int_{x_i}^{x_{i+1}} \frac{1}{\sqrt{p(x)}} dx \leq \eta (\#(X \cap I) + 2).$$

Consequently

$$\frac{\#(X \cap I)}{\mu_p(I)} \geq \frac{1}{\eta} - \frac{2}{\mu_p(I)},$$

and after taking a limit we obtain  $D_p^-(X) \geq \eta^{-1}$ .  $\square$

Proposition 6.4 shows that condition (5.8) is sharp for eventually constant parametrizing functions  $p$ . If  $X$  is defined by  $x_0 = 0$  and  $\frac{x_{i+1} - x_i}{\inf_{x \in [x_i, x_{i+1}]} \sqrt{p(x)}} = \delta > \frac{\pi}{\Omega^{1/2} - \epsilon}$ , then  $D_p^-(X) = (\Omega^{1/2} - \epsilon)/\pi < \Omega^{1/2}/\pi$  and thus cannot be a set of stable sampling by Theorem 6.3.

The remainder of the paper is devoted to the proof of Theorems 6.2 and 6.3. We follow the approach of Nitzan and Olevskii [30] who compare a discrete set of reproducing kernels to a continuous resolution of the identity in the space of bandlimited functions. Other approaches, such as the original technique of Landau [26, 25], the technique of Ortega-Cerdà and Pridhnani [31], or the approach of [16, 14], might be successful as well, but these will require additional features, such as the existence of a Riesz basis of reproducing kernels.

**6.2. Transformation to Schrödinger form.** In the first step we transform the problem from the Sturm-Liouville picture to the Schrödinger picture. This transformation enables us to use the scattering theory of the Schrödinger operator. First we describe the unitary transform that sends the operator  $A_p$  to a Schrödinger operator.

Assume  $p, p' \in AC_{loc}(\mathbb{R})$ ,  $p(x) > 0$  for all  $x \in \mathbb{R}$ . Define

$$(6.5) \quad \zeta(x) = \int_0^x p^{-1/2}(u) du,$$

so that  $\zeta(x) = \mu_p([0, x])$  for  $x > 0$  and  $\zeta(x) = -\mu_p([x, 0])$  for  $x < 0$ .

**Proposition 6.5.** *Define the Liouville transform  $U_L$  of  $f \in L^2(\mathbb{R})$  by*

$$U_L f = (p^{1/4} f) \circ \zeta^{-1}.$$

*Then  $U_L$  is a unitary operator on  $L^2(\mathbb{R})$ . It transforms the self-adjoint operator  $A_p$  to the self-adjoint Schrödinger operator  $B_q = -D^2 + q$  by conjugation:*

$$B_q = U_L A_p U_L^*, \quad \mathcal{D}(B_q) = U_L \mathcal{D}(A_p),$$

*where the potential  $q$  of  $B_q$  is*

$$(6.6) \quad q(\zeta(x)) = -p(x)^{1/4} [p \cdot (p^{-1/4})']'(x) = \frac{1}{4} p''(x) - \frac{1}{16} \frac{p'(x)^2}{p(x)}.$$

*In particular, if  $\phi$  solves  $(\tau_p - \lambda)\phi = 0$ , then  $U_L \phi$  solves  $(\tilde{\tau}_q - \lambda)\psi = 0$ .*

For a proof see [4, 10] or try a direct computation. The next lemma explains the translation from the Sturm-Liouville picture to the Schrödinger picture in more detail.

**Lemma 6.6.** *Assume  $p, p' \in AC_{loc}(\mathbb{R})$  and  $p(x) > 0$  for all  $x \in \mathbb{R}$ . Then*

- (i)  $U_L(PW_\Lambda(A_p)) = PW_\Lambda(B_q)$ .
- (ii)  $D_p^\pm(X) = D^\pm(\zeta(X))$ .
- (iii) *Let  $k$  be the reproducing kernel for  $PW_\Lambda(A_p)$  and  $h$  be the reproducing kernel for  $PW_\Lambda(B_q)$ . Then*

$$(6.7) \quad h(\zeta(x), \cdot) = p^{1/4}(x) U_L k(x, \cdot).$$

- (iv) *If  $c \leq p(x) \leq C$  for all  $x \in \mathbb{R}$ , then  $(k(x, \cdot))_{x \in X}$  is a frame (Riesz sequence) for  $PW_\Lambda(A_p)$  if and only if  $(h(\zeta(x), \cdot))_{x \in X}$  is a frame (Riesz sequence) for  $PW_\Lambda(B_q)$ .*

*Proof.* (i) Let  $c_\Lambda(A_p)$  be the spectral projection onto  $PW_\Lambda(A_p)$  and  $f \in PW_\Lambda(A_p)$ . Then

$$c_\Lambda(A_p)f = f \text{ if and only if } (U_L c_\Lambda(A_p) U_L^*) U_L f = U_L f.$$

Since by spectral calculus  $U_L c_\Lambda(A_p) U_L^* = c_\Lambda(B_q)$ , we see that  $f \in PW_\Lambda(A_p)$  if and only if  $U_L f \in PW_\Lambda(B_q)$ .

- (ii) Observe that for every interval  $[a, b] = \zeta([\alpha, \beta]) = [\zeta(\alpha), \zeta(\beta)]$  we have

$$\frac{\#(\zeta(X) \cap [a, b])}{b - a} = \frac{\#(\zeta(X) \cap \zeta([\alpha, \beta]))}{\zeta(\beta) - \zeta(\alpha)} = \frac{\#(X \cap [\alpha, \beta])}{\mu_p([\alpha, \beta])}.$$

Taking limits on both sides, we find that  $D^\pm(\zeta(X)) = D_p^\pm(X)$ .

- (iii) If  $f \in PW_\Omega(A_p)$ , then  $U_L f \in PW_\Lambda(B_q)$  by (i), and

$$(6.8) \quad U_L f(\zeta(x)) = \langle U_L f, h(\zeta(x), \cdot) \rangle.$$

On the other hand

$$\begin{aligned}
 U_L f(\zeta(x)) &= p^{1/4}(x) f(x) = p^{1/4}(x) \langle f, k(x, \cdot) \rangle \\
 (6.9) \quad &= p^{1/4}(x) \langle U_L f, U_L k(x, \cdot) \rangle = \langle U_L f, p^{1/4}(x) U_L k(x, \cdot) \rangle.
 \end{aligned}$$

The combination of (6.8) and (6.9) yields (6.7).

(iv) The image of a frame (a Riesz sequence) under an invertible operator is again a frame (a Riesz sequence). So  $\{k(x, \cdot) : x \in X\}$  is a frame (a Riesz sequence) for  $PW_\Lambda(A_p)$ , if and only if  $\{U_L k(x, \cdot) : x \in X\}$  is a frame (a Riesz sequence) for  $PW_\Lambda(B_q)$ . Since  $c \leq p(x) \leq C$ ,  $\{U_L k(x, \cdot) : x \in X\}$  is a frame (a Riesz sequence), if and only if  $\{p(x)^{1/4} U_L k(x, \cdot) : x \in X\}$  is a frame (a Riesz sequence).  $\square$

From now on we will work with the Schrödinger picture. By a slight abuse of notation we will denote the reproducing kernel for  $PW_\Lambda(B_q)$  again by the symbol  $k$ .

If  $p$  is eventually constant, then by (6.6) the potential  $q$  has compact support in some interval  $[-a, a]$ . We therefore assume that

( $MC_q$ ) the potential  $q$  is of the form (6.6) for some  $p$  satisfying (6.3).

Lemma 6.6,(iii) ,(iv) implies an equivalent formulation of the Theorems 6.2 and 6.3.

**Theorem 6.7** (Necessary density conditions in  $PW_\Lambda(B_q)$ ). *Assume that  $q$  satisfies ( $MC_q$ ). Let  $k$  be the reproducing kernel for  $PW_\Lambda(B_q)$ .*

(A) Interpolation: *If  $\{k(x, \cdot) : x \in X\}$  is a Riesz sequence in  $PW_\Lambda(B_q)$ , then*

$$D^+(X) \leq \frac{|\Lambda^{1/2}|}{\pi}.$$

(B) Sampling: *If  $\{k(x, \cdot) : x \in X\}$  is a frame for  $PW_\Lambda(B_q)$ , then*

$$D^-(X) \geq \frac{|\Lambda^{1/2}|}{\pi}.$$

Theorems 6.2 and 6.3 now follow with the translation lemma (Lemma 6.6). The proof of Theorem 6.7 will be carried out in Section 6.4

**6.3. Fundamental Lemmas.** Most of the technical work for the proof of Theorem 6.7 is coded in some lemmas on the localization and cancellation properties of the reproducing kernel for  $PW_\Lambda(B_q)$ . For the proofs we need information about the scattering theory of the Schrödinger operator.

For the spectral representation of the Schrödinger operator we substitute the spectral parameter and set  $\lambda = \omega^2$ . Thus if  $\lambda$  is in the spectral set  $\Lambda$ , then  $\omega$  is in  $\Lambda^{1/2} = \{\omega : \omega^2 \in \Lambda\}$ . This harmless, but convenient change of variables explains the appearance of the set  $\Lambda^{1/2}$  in the formulation of the density theorems.



**Proposition 6.8** ([42, 23.2],[41, 17.C]). *If  $q$  satisfies  $(MC_q)$ , then the eigenfunction equation  $(\tilde{\tau}_q - \omega^2)\phi = 0$  possesses a system of fundamental solutions of the form*

$$(6.10) \quad \Phi(\omega, x) = \begin{cases} \begin{pmatrix} e^{i\omega x} + R_1(\omega)e^{-i\omega x} \\ T(\omega)e^{-i\omega x} \end{pmatrix} & , \quad x < -a \\ \begin{pmatrix} T(\omega)e^{i\omega x} \\ e^{-i\omega x} + R_2(\omega)e^{i\omega x} \end{pmatrix} & , \quad x > a \end{cases}$$

The scattering matrix

$$(6.11) \quad \begin{pmatrix} T(\omega) & R_1(\omega) \\ R_2(\omega) & T(\omega) \end{pmatrix}$$

is unitary for all  $\omega \in \mathbb{R}^+$ , and the entries  $T, R_1, R_2$  are holomorphic in  $\omega$  for  $\omega \in \mathbb{C} \setminus \mathbb{R}_0^-$ .

The spectral measure of  $\tilde{\tau}_q$  with respect to this fundamental system is given by the matrix-valued Lebesgue measure  $\begin{pmatrix} d\omega & 0 \\ 0 & d\omega \end{pmatrix} = I_2 d\omega$ . Consequently the operator

$$(6.12) \quad \mathcal{F}_{B_q} f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \overline{\Phi(\omega, x)} dx ,$$

is unitary on  $L^2(\mathbb{R})$  and diagonalizes  $B_q$ , i.e.,

$$\mathcal{F}_{B_q} B_q \mathcal{F}_{B_q}^{-1} G(\omega) = \omega^2 G(\omega)$$

for all  $G \in L^2(\mathbb{R}, I_2 d\omega)$ . The inverse of  $\mathcal{F}_{B_q}$  is

$$(6.13) \quad \mathcal{F}_{B_q}^{-1} G(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_0^+} G(\omega) \cdot \Phi(\omega, x) d\omega$$

for  $G \in L^2(\mathbb{R}, I_2 d\omega)$ .

With this notation the reproducing kernel for  $PW_{\Lambda}(B_q)$  is simply

$$(6.14) \quad k(x, y) = k_{\Lambda}(x, y) = \frac{1}{2\pi} \int_{\Lambda^{1/2}} \Phi(\omega, x) \cdot \overline{\Phi(\omega, y)} d\omega .$$

In this case it is obvious that the kernel exists for  $|\Lambda| \leq \infty$  (see the remark after Proposition 3.3).

The following three lemmas describe several properties of the reproducing kernel. For the space of bandlimited functions  $\{f \in L^2(\mathbb{R}) : \text{supp } \hat{f} \subseteq [-\Omega, \Omega]\}$  the reproducing kernel is  $k(x, y) = \frac{\sin \Omega(x-y)}{x-y}$  and the stated estimates below are obvious. For the Paley-Wiener spaces of variable bandwidth they are highly non-trivial and, even in the model case  $(MC_q)$ , they require the full power of the scattering theory (Proposition 6.8). The following statements may also be interpreted as subtle cancellation properties of  $k$ .

**Lemma 6.9** (Weak localization). *Assume that  $(MC_q)$  holds and that  $\Lambda$  is a Borel set in  $\mathbb{R}_0^+$  with finite Lebesgue measure. Let  $k$  be the reproducing kernel for  $PW_\Lambda(B_q)$ . Then for every  $\epsilon > 0$  there is a constant  $b_\epsilon$ , such that*

$$(6.15) \quad \sup_{x \in \mathbb{R}} \int_{|y-x| > b_\epsilon} |k(x, y)|^2 dy < \epsilon^2.$$

**Lemma 6.10** (Homogenous approximation property). *Assume that  $(MC_q)$  holds and that  $\Lambda$  is a bounded Borel set in  $\mathbb{R}_0^+$ . Furthermore assume that  $X$  is a set of stable sampling for  $PW_\Lambda(B_q)$ . Then for every  $\epsilon > 0$  there is a constant  $b_\epsilon$ , such that*

$$(6.16) \quad \sup_{y \in \mathbb{R}} \sum_{\substack{x \in X \\ |x-y| > b_\epsilon}} |k(x, y)|^2 < \epsilon^2.$$

The proof of these lemmas is deferred to Section 7.

**Lemma 6.11.** *Assume that  $q$  satisfies  $(MC_q)$  with  $\text{supp } q \subseteq [-a, a]$ . Let  $I = [\alpha, \beta]$  be a large, closed interval. Then*

$$(6.17) \quad \left| \frac{1}{|I|} \int_I k(y, y) dy - \frac{|\Lambda^{1/2}|}{\pi} \right| \leq \frac{2}{\sqrt{\pi}} \frac{\|R_1 c_{\Lambda^{1/2}}\|}{|I|^{1/2}} + \frac{1}{|I|} \int_{-a}^a k(y, y) dy + \frac{2a}{\pi|I|} |\Lambda^{1/2}|.$$

As a consequence,

$$\lim_{|I| \rightarrow \infty} \frac{1}{|I|} \int_I k(y, y) dy = \frac{|\Lambda^{1/2}|}{\pi}.$$

*Proof.* After substituting the fundamental solutions (6.10) into (6.14), and using the unitarity of the scattering matrix (6.11), we obtain for  $y > a$

$$(6.18) \quad \begin{aligned} k(y, y) &= \frac{1}{2\pi} \int_{\Lambda^{1/2}} \underbrace{(|T(\omega)|^2 + |R_2(\omega)|^2)}_{=1} + 1 + 2\Re(R_2(\omega)e^{2i\omega y}) d\omega \\ &= \frac{1}{\pi} (|\Lambda^{1/2}| + \int_{\Lambda^{1/2}} \Re(R_2(\omega)e^{-2i\omega y}) d\omega). \end{aligned}$$

If  $y < -a$ , then similarly

$$k(y, y) = \frac{1}{\pi} (|\Lambda^{1/2}| + \int_{\Lambda^{1/2}} \Re(R_1(\omega)e^{-2i\omega y}) d\omega).$$

In the following we decompose the given interval  $I$  into subintervals

$$I = [\alpha, \beta] = (I \cap (-\infty, -a)) \cup (I \cap [-a, a]) \cup (I \cap (a, \infty)) = I_1 \cup I_2 \cup I_3.$$

Without loss of generality we assume that  $[-a, a] \subseteq I$  (If  $[-a, a] \not\subseteq I$ , then some of the integrals  $\int_{I_k} \dots$  are zero.) Then

$$\begin{aligned} \frac{1}{|I|} \int_{I_3} k(y, y) dy &= \frac{|I_3|}{\pi|I|} |\Lambda^{1/2}| + \frac{1}{\pi|I|} \int_{I_3} \int_{\Lambda^{1/2}} \Re(R_2(\omega) e^{2i\omega y}) d\omega dy \\ &= \frac{|I_3|}{\pi|I|} |\Lambda^{1/2}| + \frac{1}{\pi|I|} \Re \left( \int_{\Lambda^{1/2}} R_2(\omega) \int_a^\beta e^{2i\omega y} dy d\omega \right) \\ &= \frac{|I_3|}{\pi|I|} |\Lambda^{1/2}| + \frac{1}{\pi|I|} \Re \left( \int_{\Lambda^{1/2}} R_2(\omega) e^{i\omega(a+\beta)} \frac{\sin(\omega(\beta-a))}{\omega} d\omega \right), \end{aligned}$$

and, with the substitution  $u = \omega(\beta - a)$ ,

$$\begin{aligned} \frac{1}{\pi|I|} \left| \int_{\Lambda^{1/2}} R_2(\omega) e^{i\omega(a+\beta)} \frac{\sin(\omega(\beta-a))}{\omega} d\omega \right| &\leq \frac{\|R_2 c_{\Lambda^{1/2}}\|}{\pi|I|} \left( (\beta-a) \int_{(\beta-a)\Lambda^{1/2}} \frac{\sin^2 u}{u^2} du \right)^{1/2} \\ &\leq \frac{\|R_2 c_{\Lambda^{1/2}}\|}{\sqrt{\pi}} |I|^{-1/2}. \end{aligned}$$

By a similar calculation the contribution of  $I_1$  yields

$$\frac{1}{|I|} \int_{I_1} k(y, y) dy = \frac{|I_1|}{\pi|I|} |\Lambda^{1/2}| + \frac{1}{\pi|I|} \int_{I_1} \int_{\Lambda^{1/2}} \Re(R_1(\omega) e^{-2i\omega y}) d\omega dy$$

and

$$\frac{1}{\pi|I|} \left| \int_{I_1} \int_{\Lambda^{1/2}} \Re(R_1(\omega) e^{-2i\omega y}) d\omega dy \right| \leq \frac{\|R_1 c_{\Lambda^{1/2}}\|}{\sqrt{\pi}} |I|^{-1/2}.$$

After summing these contributions, we obtain

$$\begin{aligned} \left| \frac{1}{|I|} \int_I k(y, y) dy - \frac{|\Lambda^{1/2}|}{\pi} \right| &\leq \left| \frac{1}{|I|} \int_{I_1 \cup I_3} k(y, y) dy - \frac{|\Lambda^{1/2}|}{\pi} \right| + \frac{1}{|I|} \left| \int_{[-a, a]} k(y, y) dy \right| \\ (6.19) \quad &\leq \frac{1}{\sqrt{\pi}|I|^{1/2}} \left( \|R_1 c_{\Lambda^{1/2}}\| + \|R_2 c_{\Lambda^{1/2}}\| \right) + \frac{1}{|I|} \int_{-a}^a |k(y, y)| dy + \frac{2a}{\pi|I|} |\Lambda^{1/2}|. \end{aligned}$$

As the matrix (6.11) is unitary,  $|R_1(\omega)| = |R_2(\omega)|$  for all  $\omega \geq 0$ , and  $\|R_1 c_{\Lambda^{1/2}}\| = \|R_2 c_{\Lambda^{1/2}}\|$ , and (6.17) follows.  $\square$

From (6.18) and the continuity of  $k$  we extract a crucial property of the reproducing kernel  $k$ .

**Corollary 6.12.** *If  $q$  satisfies condition  $(MC_q)$ , then the diagonal of the kernel  $k$  is uniformly bounded:*

$$\sup_{y \in \mathbb{R}} k(y, y) = C_{RK} < \infty.$$

In the following lemma we gather some facts about frames and Riesz sequences in a reproducing kernel Hilbert space  $H$ .

**Lemma 6.13.** *Let  $H$  be a reproducing kernel Hilbert space with kernel  $k$ . Assume that  $\{k(x, \cdot) : x \in X\}$  is a frame for  $H$  with canonical dual frame  $\{g_x : x \in X\}$ .*

Then  $k$  and  $g_x$  satisfy the following inequalities:

$$(6.20) \quad \sum_{x \in X} k(x, y) \overline{g_x(y)} = k(y, y)$$

$$(6.21) \quad \sum_{x \in X} |g_x(y)|^2 \leq C k(y, y)$$

$$(6.22) \quad \sup_{x \in X} |\langle k(x, \cdot), g_x \rangle| \leq 1$$

$$(6.23) \quad \sup_{x \in X} \|g_x\| = C < \infty.$$

If  $\{k(x, \cdot) : x \in X\}$  is a Riesz sequence for a subspace  $V \subseteq H$  with biorthogonal basis  $\{g_x : x \in X\} \subseteq V$ , then (6.20) is replaced by the inequality

$$(6.24) \quad \sum_{x \in X} k(x, y) \overline{g_x(y)} \leq k(y, y).$$

Instead of (6.23) holds equality  $|\langle k(x, \cdot), g_x \rangle| = 1$  for all  $x \in X$ .

*Proof.* The inequality (6.24) follows from

$$\begin{aligned} \sum_{x \in X} k(x, y) \overline{g_x(y)} &= \sum_{x \in X} \langle k(x, \cdot), k(y, \cdot) \rangle \langle k(y, \cdot), g_x \rangle \\ &= \left\langle \sum_{x \in X} \langle k(y, \cdot), g_x \rangle k(x, \cdot), k(y, \cdot) \right\rangle = \langle P_V k(y, \cdot), k(y, \cdot) \rangle \\ &\leq \|k(y, \cdot)\|^2 = k(y, y). \end{aligned}$$

The proof for frames is the same (just omit the projection).

Item (6.21) follows from

$$\sum_{x \in X} |g_x(y)|^2 = \sum_{x \in X} |\langle g_x, k(y, \cdot) \rangle|^2 \leq C \|k(y, \cdot)\|^2 = C k(y, y),$$

where  $C$  is the upper frame bound for  $\{g_x : x \in X\}$ .

Item (6.22) is an immediate consequence of the minimal  $\ell^2$ -norms of the coefficients in the canonical frame expansion [8]:

$$k(x', \cdot) = \sum_{x \in X} \langle k(x', \cdot), g_x \rangle k(x, \cdot) = 1 \cdot k(x', \cdot) \quad \text{for every } x' \in X,$$

so

$$|\langle k(x', \cdot), g_{x'} \rangle|^2 \leq \sum_{x \in X} |\langle k(x', \cdot), g_x \rangle|^2 \leq 1 \quad \text{for every } x' \in X.$$

Finally (6.23) is a general fact about frames. □

**6.4. Proof of Theorem 6.7.** It is easy to see that every set of interpolation for  $PW_\Lambda(B_q)$  must be separated and that every set of stable sampling for  $PW_\Lambda(B_q)$  must be relatively separated.

(A) *Proof of the necessary density conditions for interpolation.* Assume that  $\{k(x, \cdot) : x \in X\}$  is a Riesz sequence in  $PW_\Lambda(B_q)$  with biorthogonal basis  $\{g_x : x \in X\}$ . For every closed, bounded interval  $I = [\alpha, \beta] \in \mathbb{R}$  let  $V$  be the (finite-dimensional) subspace  $V = V_I = \text{span}\{k(x, \cdot) : x \in X \cap I\}$  and  $P_V$  be the orthogonal projection from  $PW_\Lambda(A)$  onto  $V$ . Then  $\{P_V g_x : x \in X \cap I\} \subseteq V$  is the biorthogonal basis to  $\{k(x, \cdot) : x \in X \cap I\}$  and  $\|P_V g_x\| \leq \|g_x\| \leq C$  for all  $x$ . By (6.24)

$$\sum_{x \in X \cap I} k(x, y) \overline{P_V g_x(y)} \leq k(y, y).$$

We now integrate both sides over a suitably enlarged interval  $I_b = [\alpha - b, \beta + b]$  and obtain

$$(6.25) \quad \int_{I_b} \sum_{x \in X \cap I} k(x, y) \overline{P_V g_x(y)} dy \leq \int_{I_b} k(y, y) dy.$$

The biorthogonality  $\langle k(x, \cdot), P_V g_x \rangle = \langle k(x, \cdot), g_x \rangle = 1$  for  $x \in X \cap I$  implies that

$$\int_{I_b} k(x, y) \overline{P_V g_x(y)} dx = 1 - \int_{\mathbb{R} \setminus I_b} k(x, y) \overline{P_V g_x(y)} dy.$$

Now fix  $\epsilon > 0$  and let  $b = b_\epsilon$  be the weak localization constant of Lemma 6.9. If  $x \in X \cap I$  and  $y \in \mathbb{R} \setminus I_b$ , then  $|x - y| > b$ . Therefore Lemma 6.9 implies that

$$(6.26) \quad \left| \int_{\mathbb{R} \setminus I_b} k(x, y) \overline{P_V g_x(y)} dy \right|^2 \leq \|P_V k(x, \cdot)\|^2 \int_{\mathbb{R} \setminus I_b} |k(x, y)|^2 dy \leq C^2 \epsilon^2,$$

and consequently

$$\left| \sum_{x \in X \cap I} \int_{I_b} k(x, y) \overline{P_V g_x(y)} dy \right| = \left| \sum_{x \in X \cap I} \left( 1 - \int_{\mathbb{R} \setminus I_b} k(x, y) \overline{P_V g_x(y)} dy \right) \right| \geq \#(X \cap I) (1 - C\epsilon).$$

Inserting this estimate in the left-hand side of (6.25) yields the lower estimate

$$\frac{1}{|I|} \int_{I_b} k(y, y) dy \geq (1 - C\epsilon) \frac{\#(X \cap I)}{|I|},$$

whereas Lemma 6.11 leads to the upper estimate

$$\frac{1}{|I|} \int_{I_b} k(y, y) dy \leq \frac{|I_b|}{|I|} \left[ \frac{|\Lambda^{1/2}|}{\pi} + C' |I_b|^{-1/2} \right].$$

Since  $|I_b|/|I| = 1 + 2b/|I|$ , we obtain

$$\frac{\#(X \cap I)}{|I|} \leq (1 - C\epsilon)^{-1} \left[ \frac{|\Lambda^{1/2}|}{\pi} + \frac{C'}{|I|^{1/2}} \right]$$

We take the supremum over all intervals with  $|I| = r$  and then the limit  $r \rightarrow \infty$ , this yields  $D^+(X) \leq (1 - C\epsilon)^{-1} \frac{|\Lambda^{1/2}|}{\pi}$ . As  $\epsilon > 0$  was arbitrary, we have proved that  $D^+(X) \leq \frac{|\Lambda^{1/2}|}{\pi}$ .  $\square$

(B) *Proof of the necessary density conditions for sampling.* We assume that  $\{k(x, \cdot) : x \in X\}$  is a frame for  $PW_\Lambda(B_q)$  with canonical dual frame  $\{g_x : x \in X\}$ .

Since we will use Lemma 6.10, we first assume that the spectral set  $\Lambda$  is bounded.

Let  $\epsilon > 0$  be given. This time we use Lemma 6.13 (6.20) to write  $k(y, y) = \sum_{x \in X} k(x, y) \overline{g_x(y)}$  for all  $y \in \mathbb{R}$  and use this expression to estimate the averaged kernel in Lemma 6.11. By Lemma 6.11 there exists an  $r_0 = r_0(\epsilon)$  such that for all intervals  $I = [\alpha, \beta]$  of length  $r > r_0$

$$\left(\frac{|\Lambda^{1/2}|}{\pi} - \epsilon\right)|I| \leq \int_I k(y, y) dy.$$

The combination of these facts leads to

$$(6.27) \quad \left(\frac{|\Lambda^{1/2}|}{\pi} - \epsilon\right)|I| \leq \int_I k(y, y) dy = \left| \int_I \sum_{x \in X} k(x, y) \overline{g_x(y)} dy \right|.$$

Let  $b = b_\epsilon$  be larger than both constants  $b_\epsilon$  from Lemmas 6.9 and 6.10 and set  $I_- = [\alpha + b, \beta - b]$  and  $I_+ = [\alpha - b, \beta + b]$ . We partition  $X$  accordingly and write

$$\begin{aligned} \sum_{x \in X} k(x, y) \overline{g_x(y)} &= \left( \sum_{x \in X \cap I_-} + \sum_{x \in X \cap (\mathbb{R} \setminus I_+)} + \sum_{x \in X \cap (I_+ \setminus I_-)} \right) k(x, y) \overline{g_x(y)} \\ &= A_1(y) + A_2(y) + A_3(y) \end{aligned}$$

*Estimate of  $\int_I A_2$ .* Note that  $y \in I$  and  $x \in X \setminus I_+$  implies that  $|x - y| > b$ . Lemma 6.10 asserts that  $\sum_{x \in X \setminus I_+} |k(x, y)|^2 < \epsilon^2$ . Consequently, using also (6.21), we obtain

$$\begin{aligned} \left| \int_I A_2(y) dy \right| &\leq \int_I \left( \sum_{x \in X \cap (\mathbb{R} \setminus I_+)} |k(x, y)|^2 \right)^{1/2} \left( \sum_{x \in X} |g_x(y)|^2 \right)^{1/2} dy \\ &\leq C \epsilon \sup_{y \in \mathbb{R}} k(y, y)^{1/2} |I|. \end{aligned}$$

Since the diagonal of the kernel  $k$  is uniformly bounded by Corollary 6.12, the final estimate for  $A_2$  is

$$(6.28) \quad \left| \int_I A_2(y) dy \right| \leq C_2 \epsilon |I|.$$

*Estimate of  $\int_I A_3$ .* For the third term observe that

$$(6.29) \quad \int_I |A_3(y)| dy \leq \sum_{x \in X \cap (I_+ \setminus I_-)} \int_{\mathbb{R}} |k(x, y)| |g_x(y)| dy \leq \sum_{x \in X \cap (I_+ \setminus I_-)} \|k(x, \cdot)\| \|g_x\|.$$

Since  $X$  is relatively separated with covering constant  $n_0 = \max_{c \in \mathbb{R}} \#(X \cap [c, c + 1])$ , this sum contains at most  $(|I_+ \setminus I_-| + 1)n_0 \leq (2b + 1)n_0$  terms. Using the boundedness of the diagonal of  $k$  from Corollary 6.12 and of the canonical dual frame, the final estimate for  $A_3$  is

$$(6.30) \quad \int_I |A_3(y)| dy \leq C C_{RK} (2b + 1) n_0 = C_3.$$

Here  $C_3$  is a constant depending on  $\epsilon$  via  $b = b_\epsilon$ , but  $C_3$  is independent of  $I$ .

*Estimate of  $\int_I A_1$ .* Next we estimate  $|\int_I A_1(y)dy|$ . Since  $\int_I = \int_{\mathbb{R}} - \int_{\mathbb{R} \setminus I}$  and  $|\langle k(x, \cdot), g_x \rangle| \leq 1$  by Lemma 6.13, we obtain

$$(6.31) \quad \left| \int_I k(x, y) \overline{g_x(y)} dy \right| \leq \left| \int_{\mathbb{R}} k(x, y) \overline{g_x(y)} dy \right| + \left| \int_{\mathbb{R} \setminus I} k(x, y) \overline{g_x(y)} dy \right|$$

$$(6.32) \quad \leq 1 + \left( \int_{\mathbb{R} \setminus I} |k(x, y)|^2 dy \right)^{1/2} \|g_x\|.$$

From  $x \in X \cap I_-$  and  $y \in \mathbb{R} \setminus I$  it follows that  $|x - y| > b$ , therefore by Lemma 6.9 a single term contributing to  $A_1$  is majorized by

$$\left| \int_I k(x, y) \overline{g_x(y)} dy \right| \leq 1 + C_1 \epsilon.$$

This estimate implies

$$(6.33) \quad \left| \int_I A_1(y) dy \right| \leq \sum_{x \in X \cap I_-} \left| \int_I k(x, y) \overline{g_x(y)} dy \right| \leq (1 + C_1 \epsilon) \#(X \cap I_-).$$

Combining the estimates for  $A_1, A_2, A_3$ , we obtain

$$\begin{aligned} \left( \frac{|\Lambda^{1/2}|}{\pi} - \epsilon \right) |I| &\leq \left| \int_I A_1(y) dy \right| + \left| \int_I A_2(y) dy \right| + \left| \int_I A_3(y) dy \right| \\ &\leq (1 + C_1 \epsilon) \#(X \cap I) + C_2 \epsilon |I| + C_3. \end{aligned}$$

Therefore

$$(6.34) \quad \frac{\#(X \cap I)}{|I|} \geq (1 + C_1 \epsilon)^{-1} \left( \frac{|\Lambda^{1/2}|}{\pi} - \epsilon - C_2 \epsilon - \frac{C_3}{|I|} \right).$$

Now take the infimum over  $|I| = r$  and let  $r$  tend to  $\infty$ . Again, since  $\epsilon > 0$  is arbitrary, we conclude that  $D^-(X) \geq \frac{|\Lambda^{1/2}|}{\pi}$ .

So far we have proved the necessary density condition  $D^-(X) \geq \frac{|\Lambda^{1/2}|}{\pi}$  under the assumption that the spectral set  $\Lambda$  is bounded.

Now let  $\Lambda \subseteq \mathbb{R}^+$  be an arbitrary set of finite measure and assume that  $X$  is a set of stable sampling for  $PW_{\Lambda}(B_q)$ . Let  $\Omega > 0$ . Then  $\Lambda \cap [0, \Omega]$  is bounded and the Paley-Wiener space  $PW_{\Lambda \cap [0, \Omega]}(B_q)$  is a closed subspace of  $PW_{\Lambda}(B_q)$ . In particular, every set of stable sampling for  $PW_{\Lambda}(B_q)$  is a set of stable sampling for  $PW_{\Lambda \cap [0, \Omega]}(B_q)$ . From the main part of the proof we conclude that

$$D^-(X) \geq \frac{|\Lambda^{1/2} \cap [0, \Omega^{1/2}]|}{\pi}.$$

Since  $\Omega > 0$  was arbitrary, we conclude that  $D^-(X) \geq |\Lambda^{1/2}|/\pi$ . Thus this necessary condition holds for arbitrary spectral sets of finite measure.  $\square$



## 7. LOCALIZATION AND CANCELLATION PROPERTIES OF THE REPRODUCING KERNEL

In this section we prove the decisive Lemmas 6.9 and 6.10.

*Proof of weak localization — Lemma 6.9.* Since  $|k(x, y)| = |k(y, x)|$ , we will show that there exists a  $b = b_\epsilon$  such that  $\int_{|x-y|>b} |k(x, y)|^2 dx < \epsilon^2$  for all  $y \in \mathbb{R}$ .

We distinguish several cases.

**Case I:  $|y| \leq a$ .** Since  $|x - y| \geq |x| - |y| \geq |x| - a$ , it suffices to show that there is a constant  $b$ , such that  $\int_{|x|>b} |k(x, y)|^2 dx < \epsilon$ . We use a compactness argument for this case.

We first verify that  $y \mapsto k(\cdot, y)$  is a continuous mapping from  $[-a, a]$  to  $L^2(\mathbb{R})$ , so the set  $\{k(\cdot, y) : |y| \leq a\}$  is compact in  $L^2(\mathbb{R})$ . Then by the Kolmogorov-Riesz theorem (e.g., [17, 43]) there is a constant  $b_\epsilon$  such that

$$\|k(\cdot, y) c_{|\cdot|>b_\epsilon}\| < \epsilon \text{ for all } y \in [-a, a].$$

To verify the continuity of  $y \mapsto k(\cdot, y)$  we use the dual characterization of the norm as follows:

$$\begin{aligned} \|k(\cdot, y) - k(\cdot, y')\| &= \sup\{|\langle k(\cdot, y) - k(\cdot, y'), f \rangle| : f \in PW_\Lambda(B_q), \|f\| \leq 1\} \\ &= \sup\{|f(y) - f(y')| : f \in PW_\Lambda(B_q), \|f\| \leq 1\}, \end{aligned}$$

Using the representation formula of Lemma 3.2, we obtain

$$\begin{aligned} |f(y) - f(y')| &= \left| \int_{\Lambda^{1/2}} \mathcal{F}_{B_q} f(\omega) \cdot [\Phi(\omega, y) - \Phi(\omega, y')] d\omega \right| \\ &\leq \int_{\Lambda^{1/2}} |\mathcal{F}_{B_q} f(\omega)| |\Phi(\omega, y) - \Phi(\omega, y')| d\omega. \end{aligned}$$

An argument of Stolz [38, Thm.6] (see also [36, 3.1]) asserts that  $\Phi$  is uniformly bounded on  $[-a, a] \times \mathbb{R}^+$ . (Actually, the theorem states the boundedness of  $\Phi(\lambda, \cdot)$ , but the proof verifies boundedness in the spectral variable as well.)

Set  $C_\Phi = \sup_{x \in \mathbb{R}, \lambda \in \mathbb{R}^+} |\Phi(\lambda, x)|$  and choose  $\epsilon > 0$ . Then there is a number  $u > 0$  such that  $|\Lambda^{1/2} \setminus [0, u]| < \left(\frac{\epsilon}{2C_\Phi}\right)^2$ . Since  $\Phi$  is uniformly continuous on the closure of  $[-a, a] \times \Lambda^{1/2} \cap [0, u]$ , we can choose  $\delta > 0$  such that  $|\Phi(\omega, y) - \Phi(\omega, y')| < \epsilon$  for all  $\omega \in \Lambda^{1/2} \cap [0, u]$  and  $|y - y'| < \delta$ . Then

$$\begin{aligned} \int_{\Lambda^{1/2} \cap [0, u]} |\mathcal{F}_{B_q} f(\omega)| |\Phi(\omega, y) - \Phi(\omega, y')| d\omega &\leq \epsilon \|\mathcal{F}_{B_q} f\|_{L^1(\Lambda^{1/2})} \\ &\leq \epsilon C_{\Lambda^{1/2}} \|\mathcal{F}_{B_q} f\|_{L^2(\Lambda^{1/2})} = \epsilon C_{\Lambda^{1/2}} \|f\|_{L^2(\mathbb{R})}, \end{aligned}$$

because  $L^2(\Lambda^{1/2}, I_2 d\omega) \subseteq L^1(\Lambda^{1/2}, I_2 d\omega)$  and the spectral transform is unitary. On the other hand

$$\begin{aligned} \int_{\Lambda^{1/2} \setminus [0, u]} |\mathcal{F}_{B_q} f(\omega)| |\Phi(\omega, y) - \Phi(\omega, y')| d\omega &\leq 2C_\Phi \int_{\Lambda^{1/2} \setminus [0, u]} |\mathcal{F}_{B_q} f(\omega)| d\omega \\ &\leq 2C_\Phi \|\mathcal{F}_{B_q} f\|_{L^2(\Lambda^{1/2})} (|\Lambda^{1/2} \setminus [0, u]|)^{1/2} \\ &\leq \epsilon \|\mathcal{F}_{B_q} f\|_{L^2(\Lambda^{1/2})}, \end{aligned}$$

so we obtain for  $|y - y'| < \delta$

$$|f(y) - f(y')| < C\epsilon \|f\|.$$

Taking the supremum over all  $f$  in the unit ball of  $PW_\Lambda(B_p)$  we obtain the desired continuity of  $y \mapsto k(y, \cdot)$ .

**Case II:  $y > a$ .** We split the integral into three parts as follows:

$$\begin{aligned} \int_{|x-y|>b} |k(x, y)|^2 dx &= \int_{\substack{|x-y|>b \\ |x|\leq a}} |k(x, y)|^2 dx + \int_{\substack{|x-y|>b \\ x>a}} |k(x, y)|^2 dx \\ &\quad + \int_{\substack{|x-y|>b \\ x<-a}} |k(x, y)|^2 dx = A + B + C, \end{aligned}$$

and estimate each integral separately.

To estimate  $A$ , it is sufficient to find a value  $b_0$  large enough, such that

$$(7.1) \quad \int_{|x|\leq a} |k(x, y)|^2 dx < \epsilon \quad \text{for all } |y| \geq b_0.$$

By a straightforward calculation

$$\begin{aligned} \int_{|x|\leq a} |k(x, y)|^2 dx &= \int_{|x|\leq a} \left( \int_{\Lambda^{1/2}} \Phi(\omega, x) \cdot \overline{\Phi(\omega, y)} d\omega \int_{\Lambda^{1/2}} \Phi(\mu, x) \cdot \overline{\Phi(\mu, y)} d\mu \right) dx \\ &= \sum_{i,k=1}^2 \iint_{\Lambda^{1/2} \times \Lambda^{1/2}} \left( \int_{|x|\leq a} \Phi_i(\omega, x) \overline{\Phi_k(\mu, x)} dx \right) \cdot \overline{\Phi_i(\omega, y)} \Phi_k(\mu, y) d\mu d\omega \\ &= \sum_{i,k=1}^2 \iint_{\Lambda^{1/2} \times \Lambda^{1/2}} \Psi_{i,k}(\omega, \mu) \cdot \overline{\Phi_i(\omega, y)} \Phi_k(\mu, y) d\mu d\omega. \end{aligned}$$

Here the functions  $\Psi_{i,k}$  are continuous in  $\omega$  and  $\mu$ . By (6.10) for  $|y| \geq a$ , the products  $\overline{\Phi_i(\omega, y)} \Phi_k(\mu, y)$  are of the form

$$(7.2) \quad \alpha_{ik}(\omega, \mu) e^{\pm i(\omega - \mu)y} + \beta_{ik}(\omega, \mu) e^{\pm i(\omega + \mu)y}$$

with smooth coefficients  $\alpha_{ik}, \beta_{ik}$ . Consequently, we may interpret the map  $y \rightarrow \int_{|x|\leq a} |k(x, y)|^2 dy$  as a sum of two-dimensional Fourier transforms of continuous functions on  $\Lambda^{1/2} \times \Lambda^{1/2}$ . By the Riemann-Lebesgue Lemma  $\lim_{y \rightarrow \infty} \int_{|x|\leq a} |k(x, y)|^2 dy = 0$  and (7.1) is proved.

To estimate the term  $B$ , we first obtain an explicit expression for  $k(x, y)$  in terms of the scattering coefficients. Since  $x, y > a$ , the scattering relations (6.10) yield

$$\begin{aligned} \Phi(\omega, x) \cdot \overline{\Phi(\omega, y)} &= e^{i\omega(x-y)} \left( \underbrace{|T(\omega)|^2 + |R_2(\omega)|^2}_{=1} + e^{-2i\omega(x-y)} + R_2(\omega)e^{2i\omega y} + \overline{R_2(\omega)}e^{-2i\omega x} \right) \\ &= e^{i\omega(x-y)} + e^{-i\omega(x-y)} + R_2(\omega)e^{i\omega(x+y)} + \overline{R_2(\omega)}e^{-i\omega(x+y)}. \end{aligned}$$

After integrating the last expression over  $\Lambda^{1/2}$ , we obtain

$$(7.3) \quad k(x, y) = \mathcal{F}(c_{\Lambda^{1/2}})(y-x) + \mathcal{F}(c_{\Lambda^{1/2}})(x-y) + \mathcal{F}(R_2 c_{\Lambda^{1/2}})(-x-y) + \mathcal{F}(\overline{R_2} c_{\Lambda^{1/2}})(x+y).$$

Since  $c_{\Lambda^{1/2}}$  and  $R_2 c_{\Lambda^{1/2}}$  are in  $L^2(\mathbb{R})$ , so are their Fourier transforms. Thus there exists a constant  $b_1$ , such that

$$\int_{|z| > b_1} \left( |\mathcal{F}c_{\Lambda^{1/2}}(z)|^2 + |\mathcal{F}(R_2 c_{\Lambda^{1/2}})(z)|^2 \right) dz < \epsilon^2.$$

Consequently for  $x + y > |y - x| \geq b_1$  and  $|x| \geq a$ , we obtain

$$B = \int_{|y-x| > b_1, |x| > a} |k(x, y)|^2 dx < \epsilon^2.$$

To estimate  $C$ , where  $x < -a$  and  $y > a$ , we use again the unitarity of the scattering matrix and obtain

$$(7.4) \quad \Phi(\omega, x) \cdot \overline{\Phi(\omega, y)} = e^{i\omega(x-y)} \overline{T(\omega)} + e^{-i\omega(x-y)} T(\omega) + e^{-i\omega(x+y)} \underbrace{\left( R_1(\omega) \overline{T(\omega)} + T(\omega) \overline{R_2(\omega)} \right)}_{=0}.$$

Thus the kernel is of the form

$$(7.5) \quad k(x, y) = \mathcal{F}(\overline{T} c_{\Lambda^{1/2}})(y-x) + \mathcal{F}(T c_{\Lambda^{1/2}})(x-y),$$

and again there exists a  $b_2$  such that  $\int_{|y-x| \geq b_2} |k(x, y)|^2 dx < \epsilon^2$ .

By combination of these cases and adjusting the choice of  $b$ ,  $b = \max(b_0, b_1, b_2)$ , the statement follows when  $y > a$ .

**Case III:  $y < -a$ .** This case is treated in complete analogy to the case  $y > a$  by using the remaining scattering relations (6.10).  $\square$

*Remark.* At this point we must alert the reader to the miracle happening in (7.4). The unitarity of the scattering matrix implies that the coefficient of  $e^{i\omega(x+y)}$  vanishes. If this were not the case, we would have no control over the size of  $\int_{|x-y| \geq b_2} |k(x, y)|^2 dx$ , and the whole proof would break down. It is this seemingly little detail that made us favor the Schrödinger picture over the Sturm-Liouville picture.

*Proof of the homogenous approximation property — Lemma 6.10.* We must show that there exists  $b > 0$  such that  $\sum_{x \in X: |x-y| > b} |k(x, y)|^2 < \epsilon^2$  for all  $y \in \mathbb{R}$ . This is the discrete analogue of Lemma 6.9, and its proof is roughly parallel to the one of Lemma 6.9.

**Case I:  $|y| \leq a$ .** As  $X$  is a set of stable sampling, the mapping  $f \mapsto (f(x))_{x \in X}$  is continuous from  $PW_\Lambda(B_q)$  to  $\ell^2(X)$ , and thus maps compact sets in  $PW_\Lambda(B_q)$  to compact sets in  $\ell^2(X)$ . Applying this remark to the compact set  $\{k(\cdot, y) : |y| \leq a\}$  (as shown in the proof of Lemma 6.9, Case I), we see that the set of samples  $\{(k(x, y))_{x \in X} : |y| \leq a\}$  is compact in  $\ell^2(X)$ .

The version of the Kolmogorov-Riesz theorem for sequences implies that for every  $\epsilon > 0$  there exists a number  $b = b_\epsilon$  such that

$$\sup_{|y| \leq a} \sum_{\substack{x \in X \\ |x| > b}} |k(x, y)|^2 < \epsilon,$$

as claimed.

**Case II:  $y > a$ .**

We split the sum into three parts:

$$\sum_{x \in X: |x-y| > b} |k(x, y)|^2 = \sum_{\substack{|x-y| > b \\ |x| \leq a}} |k(x, y)|^2 + \sum_{\substack{|x-y| > b \\ x > a}} |k(x, y)|^2 + \sum_{\substack{|x-y| > b \\ x < -a}} |k(x, y)|^2 = A + B + C.$$

*Estimate of A.* We seek  $b_0$  large enough, such that

$$\sum_{\substack{x \in X \\ |x| \leq a}} |k(x, y)|^2 < \epsilon \quad \text{for } |y| > b_0 > a$$

As in the proof of the parallel case of Lemma 6.9 we obtain

$$\begin{aligned} A &= \sum_{\substack{x \in X \\ |x| \leq a}} |k(x, y)|^2 dx = \sum_{i,k=1}^2 \iint_{\Lambda^{1/2} \times \Lambda^{1/2}} \left( \sum_{\substack{x \in X \\ |x| \leq a}} \Phi_i(\omega, x) \overline{\Phi_k(\mu, x)} \right) \overline{\Phi_i(\omega, y)} \Phi_k(\mu, y) d\mu d\omega \\ &= \sum_{i,k=1}^2 \iint_{\Lambda^{1/2} \times \Lambda^{1/2}} \Psi_{ik}(\omega, \mu) \overline{\Phi_i(\omega, y)} \Phi_k(\mu, y) d\mu d\omega, \end{aligned}$$

where we denote the inner sum by  $\Psi_{ik}(\omega, \mu) = \sum_{\substack{x \in X \\ |x| \leq a}} \Phi_i(\omega, x) \overline{\Phi_k(\mu, x)}$ . Since  $X$  is relatively separated,  $\#(X \cap [-a, a])$  is finite. Furthermore, the set of eigenfunctions is uniformly bounded on the compact set  $\Lambda^{1/2} \times [-a, a]$ , therefore  $\Psi_{ik}$  is continuous and bounded on  $\Lambda^{1/2} \times \Lambda^{1/2}$ .

Next, as in the proof of Lemma 6.9, the mapping  $y \mapsto \sum_{x \in X: |x| \leq a} |k(x, y)|^2$  is the two-dimensional Fourier transform of a bounded continuous function, and by the Riemann-Lebesgue Lemma we obtain  $\lim_{y \rightarrow \infty} \sum_{x \in X: |x| \leq a} |k(x, y)|^2 = 0$ . Thus  $|A| < \epsilon$  for all  $y$  sufficiently large.

*Estimate of B and C.* For the estimate of  $B$  and  $C$  we use the formulas for the kernel (7.3) (for  $x > a$ ) and (7.5) (for  $x < -a$ ). In both cases the kernel is a sum of Fourier transform of the scattering coefficients restricted to  $\Lambda^{1/2}$ , i.e., of  $c_{\Lambda^{1/2}}, Tc_{\Lambda^{1/2}}, R_1c_{\Lambda^{1/2}}$ , and  $R_2c_{\Lambda^{1/2}}$ . Since  $\Lambda^{1/2}$  is assumed to be bounded, the function  $x \mapsto k(x, y)$  is thus the restriction of a classical bandlimited function to one of the intervals  $[a, \infty)$  or  $(-\infty, -a]$ .

We use a local version of the Plancherel-Polya-Theorem from [16, Lemma 1]. We set  $f^\sharp(x) = \sup_{|y-x| \leq 1} |f(y)|$  and note that if  $f \in L^2(\mathbb{R})$  is bandlimited with  $\text{supp } \hat{f} \subseteq [\Omega, \Omega]$ , say, then  $f^\sharp \in L^2(\mathbb{R})$ . If  $X$  is separated with separation  $\min_{x, x' \in X} |x - x'| \geq 1$ , then

$$(7.6) \quad \sum_{x \in X, |x| \geq b} |f(x)|^2 \leq \int_{|x| \geq b-1} |f^\sharp(x)|^2 dx.$$

If  $X$  is relatively separated, then this inequality holds with the constant  $n_0 = \max_{c \in \mathbb{R}} \#(X \cap [c, c+1])$  on the right hand side. We now apply (7.6) to the functions  $x \mapsto \mathcal{F}c_{\Lambda^{1/2}}(x-y)$  and  $x \mapsto \mathcal{F}(R_2c_{\Lambda^{1/2}})(x+y)$  and obtain

$$\sum_{\substack{x \in X, |x-y| > b \\ x > a}} |\mathcal{F}c_{\Lambda^{1/2}}(x-y)|^2 \leq n_0 \int_{|x-y| \geq b-1} |(\mathcal{F}c_{\Lambda^{1/2}})^\sharp(x-y)|^2 dx = n_0 \int_{|z| \geq b-1} |(\mathcal{F}c_{\Lambda^{1/2}})^\sharp(z)|^2 dz.$$

and for  $y > a$  also

$$\sum_{\substack{x \in X: |x-y| > b \\ x > a}} |\mathcal{F}(R_2c_{\Lambda^{1/2}})(x-y)|^2 \leq n_0 \int_{|z| \geq b-1} |\mathcal{F}(R_2c_{\Lambda^{1/2}})^\sharp(z)|^2 dz.$$

Consequently,

$$B = \sum_{\substack{|x-y| > b \\ x > a}} |k(x, y)|^2 \leq 4n_0 \int_{|z| \geq b-1} (|(\mathcal{F}c_{\Lambda^{1/2}})^\sharp(z)|^2 + |(\mathcal{F}(R_2c_{\Lambda^{1/2}})^\sharp(z)|^2) dz < \epsilon,$$

for  $b$  large enough. Likewise, for large  $b$

$$C = \sum_{\substack{|x-y| > b \\ x < -a}} |k(x, y)|^2 \leq 4n_0 \int_{|z| \geq b-1} |(\mathcal{F}(Tc_{\Lambda^{1/2}})^\sharp(z)|^2 dz < \epsilon$$

**Case III:  $y < -a$ .** This case is symmetric to Case II and settled with the same argument.  $\square$

## 8. SUMMARY AND OUTLOOK

In this work we have argued that the spectral subspaces of a Sturm-Liouville operator  $f \mapsto -(pf')'$  on  $L^2(\mathbb{R})$  with a positive parametrizing function  $p$  may serve as a model for functions of variable bandwidth. Our results strongly support the intuition that the quantity  $p(x)^{-1/2}$  is a measure for the local bandwidth of such a function. This intuition is backed up by sampling theorems (with only minimal assumptions on  $p$ ), and by necessary density conditions for sampling and for interpolation (for the model case of eventually constant  $p$ ).

Clearly the project of variable bandwidth has a much bigger scope than can be treated in a single paper. The notion of variable bandwidth requires much more and deeper investigations and obviously raises a multitude of new questions both in sampling theory and also about the fine spectral properties of Sturm-Liouville operators. Let us sketch a few directions (some of which we plan to address in subsequent publications, and some of which we have no concrete ideas about).

(a) Clearly the model case of an eventually constant bandwidth parametrization  $p$  is quite restrictive. It seems that a version of Theorem 6.7 can be proved under the assumption that  $p$  is *asymptotically constant*, i.e.,  $|p(x)^{-1} - p_{\pm}^{-1}| = \mathcal{O}(|x|^{-\alpha})$  as  $x \rightarrow \pm\infty$  for some  $\alpha > 1$ . This case, however, requires much more spectral theory of Sturm Liouville operators, which according to [42] is “decidedly more complicated”.

(b) In view of the classical results of Beurling [3] one may conjecture that the density condition  $D_p^-(X) > \Omega^{1/2}/\pi$  is sufficient for  $X$  to be a set of sampling for  $PW_{[0,\Omega]}(A_p)$ , at least for reasonable  $p$ . At this time it is not clear how to replace the maximum gap condition in Theorem 5.2 by the average density of Beurling, because the functions in  $PW_{[0,\Omega]}(A_p)$  are no longer entire.

(c) In the special case  $p(x) = p_-$  for  $x \leq 0$  and  $p(x) = p_+$  for  $x > 0$  we have found a set of sampling and interpolation, equivalently, an orthonormal basis of reproducing kernels. Is there a Riesz basis of reproducing kernels in  $PW_{\Lambda}(A_p)$  for more general parametrizing functions  $p$ ? This problem is hard even for  $p \equiv 1$  and disconnected spectral sets  $\Lambda$ . See [24] for a recent breakthrough.

(d) Spectral perturbation theory: How are the Paley-Wiener spaces  $PW_{\Lambda}(A_{p_1})$  and  $PW_{\Lambda}(A_{p_2})$  related when  $p_1$  and  $p_2$  are close in some sense?

(e) Clearly all questions may be treated in the multivariate setting by considering a strongly elliptic second order differential operator  $f \rightarrow -\nabla(M\nabla f)$  for some matrix-valued function  $x \rightarrow M(x)$ . In this case only the existence of frames of reproducing kernels is known from the general work of Pesenson and Zayed [32, 34] (by sampling densely enough), but all quantitative questions about necessary and sufficient conditions for sampling are wide open. Likewise, the connection of the spectral subspaces to a local bandwidth is far from transparent. In higher dimensions we expect the geometry associated to elliptic differential operators to play a more visible and prominent role. In Sections 5 and 6 the explicit metric  $d(y, z) = |\int_y^z p(x)^{-1/2} dx|$  played an important role, in higher dimensions analogous metrics are known as Carnot-Carathéodory metrics, see, e.g., [27, 28]. We expect these to appear in the correct definition of a Beurling density and in the formulation of sampling results.

(f) Computational aspects: the ultimate goal would be the use of variable bandwidth for adaptive signal reconstruction. The idea is choose the local bandwidth according to the local sampling density and then reconstruct a function in a space of variable bandwidth. Given a nonuniform sampling set  $X = \{x_j\}$  and samples  $y_j = f(x_j)$ , we would like to proceed as follows: (i) find a bandwidth parametrizing function  $p$  such that the maximum gap condition (5.8) is satisfied. (ii) Construct a function in  $PW_{[0,\Omega]}(A_p)$  from the data  $(X, f(X))$  by means of an algorithm based on Theorem 5.3. This may seem hopeless for general parametrizing functions  $p$ , because any procedure requires the knowledge of the reproducing kernel. The discussion of Section 4 shows that at least for piecewise constant  $p$  the sampling theory can be made more explicit. Therefore this idea carries some potential for the numerical realization.

## APPENDIX A. COMPUTATION OF THE SPECTRAL MEASURE

**Spectral measure.** For the explicit construction of the spectral measure  $\rho$  assume  $\tau$  is LP at  $\pm\infty$ , and denote the unique solution of  $(\tau - z)\phi = 0$ ,  $z \notin \mathbb{R}$ , that lie left (right) in  $L^2(\mathbb{R})$  by  $\phi_-(z, \cdot)$  (by  $\phi_+(z, \cdot)$ ). Then the resolvent  $R_z(A) = (A - z)^{-1}$  of the self-adjoint realization  $A$  of  $\tau$  is the integral operator

$$(A.1) \quad R_z(A)g(x) = \frac{1}{W_p(\phi_+, \phi_-)(z)} \left( \phi_+(z, x) \int_{-\infty}^x \phi_-(z, u)g(u) du + \phi_-(z, x) \int_x^{\infty} \phi_+(z, u)g(u) du \right),$$

where  $W_p(f, g)(z) = (pf'g - fpg')(z, x)$  is the generalized Wronski determinant. Note that  $W_p(\phi_+, \phi_-)$  is independent of  $x$  for the solutions of  $(\tau - z)\phi = 0$  [42, 13.21], [39, Eq. (9.6)].

Assume that the components of  $\Phi(z, \cdot)$  form a fundamental system of solutions of  $(\tau - z)\phi = 0$  that depend continuously on  $z$  in a complex neighborhood  $Q$  of the interval  $(\alpha, \beta) \subseteq \mathbb{R}$ . Then there exist  $2 \times 2$  matrices  $m^\pm(z)$  for  $z \in Q \cap (\mathbb{C} \setminus \sigma(A))$ , such that the integral kernel  $r_z$  of the resolvent  $R_z(A)$  can be written as

$$(A.2) \quad r_z(x, y) = \begin{cases} \overline{\Phi(\bar{z}, x)} \cdot m^+(z) \Phi(z, y), & y \leq x, \\ \overline{\Phi(\bar{z}, x)} \cdot m^-(z) \Phi(z, y), & y > x. \end{cases}$$

For an interval  $(\gamma, \lambda] \subseteq (\alpha, \beta)$  the spectral measure is given by the *Weyl-Titchmarsh-Kodaira formula*

$$(A.3) \quad \rho((\gamma, \lambda]) = \frac{1}{2\pi i} \lim_{\delta \searrow 0} \lim_{\epsilon \searrow 0} \int_{\gamma+\delta}^{\lambda+\delta} (m^\pm(t + i\epsilon) - m^\pm(t - i\epsilon)) dt.$$

See e.g [42, 14.5], [9, XII, 5.18] for a proof.

We now compute the spectral measure for  $A_p$  studied in Section 4.

Set  $\phi_1 = \phi_+$ ,  $\phi_2 = \phi_-$  and

$$\begin{aligned} a_1 &= \frac{1}{2} \left( 1 + \sqrt{\frac{p_+}{p_-}} \right), & b_1 &= \frac{1}{2} \left( 1 - \sqrt{\frac{p_+}{p_-}} \right) \\ a_2 &= \frac{1}{2} \left( 1 + \sqrt{\frac{p_-}{p_+}} \right), & b_2 &= \frac{1}{2} \left( 1 - \sqrt{\frac{p_-}{p_+}} \right). \end{aligned}$$

Then

$$W_p(\phi_1, \phi_1) = -i\sqrt{z}(\sqrt{p_+} + \sqrt{p_-}).$$

For  $y \leq x$  the resolvent kernel can be written as

$$\begin{aligned} r_z(x, y) &= \frac{1}{W_p(\phi_1, \phi_2)} \phi_1(z, x) \phi_2(z, y) \\ &= \sum_{j,k=1}^2 m_{jk}^+ \overline{\phi_j(\bar{z}, x)} \phi_k(z, y) \end{aligned}$$

If  $x > 0$ ,

$$\phi_1(z, x) = \frac{1}{a_2} \left( \overline{\phi_2(\bar{z}, x)} - \overline{b_2} \phi_1(\bar{z}, x) \right),$$



so

$$r_z(x, y) = \frac{1}{W_p(\phi_1, \phi_2) \overline{a_2}} \left( -\overline{b_2} \overline{\phi_1(\bar{z}, x)} \phi_2(z, y) + \overline{\phi_2(\bar{z}, x)} \phi_2(z, y) \right)$$

which yields for the matrix  $m^+$

$$m^+(z) = \frac{1}{W_p(\phi_1, \phi_2) \overline{a_2}} \begin{pmatrix} 0 & * \\ 0 & 1 \end{pmatrix}$$

By a similar calculation for  $x < 0$ ,

$$m^+(\bar{z}) = \frac{1}{\overline{W_p(\phi_1, \phi_2) a_1}} \begin{pmatrix} 1 & ** \\ 0 & 0 \end{pmatrix},$$

and so

$$m^+(\lambda + 0i) - m^+(\lambda - 0i) = \frac{i}{\sqrt{\lambda}(\sqrt{p_+} + \sqrt{p_-})^2} \begin{pmatrix} 2\sqrt{p_-} & *** \\ 0 & 2\sqrt{p_+} \end{pmatrix}.$$

## APPENDIX B. TIME WARPING

We discuss briefly how time-warping is related to our approach with spectral subspaces. Let  $p$  be a parametrizing function with  $0 < c \leq p(x) \leq C < \infty$  and consider the differential expression  $f \rightarrow -ipf'$ . By choosing the correct measure on  $\mathbb{R}$  and an appropriate domain, we obtain the self-adjoint operator

$$B_p = -ip \frac{d}{dx}, \quad \mathcal{D}(B_p) = \left\{ f \in L^2\left(\mathbb{R}, \frac{dx}{p(x)}\right) : B_p f \in L^2\left(\mathbb{R}, \frac{dx}{p(x)}\right) \right\}$$

on  $L^2\left(\mathbb{R}, \frac{dx}{p(x)}\right)$  with corresponding spectral projections  $c_\Lambda(B_p)$  for  $\Lambda \subseteq \mathbb{R}$ . Thus a function  $f \in L^2\left(\mathbb{R}, dx/p(x)\right)$  is  $B_p$ -bandlimited with spectral set  $\Lambda$ , in short  $f \in PW_\Lambda(B_p)$ , if  $f = c_\Lambda(B_p)f$ . In this case bandlimited functions possess the following explicit description.

**Proposition B.1.** *Set  $\eta(x) = \int_0^x \frac{1}{p(t)} dt$ . Then  $f \in PW_\Lambda(B_p)$ , if and only if there exists  $F \in L^2(\mathbb{R})$  with  $\text{supp } F \subseteq \Lambda$ , such that*

$$(B.1) \quad f(x) = \int_\Lambda F(\lambda) e^{i\lambda\eta^{-1}(x)} d\lambda = (\mathcal{F}^{-1}F)(\eta^{-1}(x)).$$

Thus  $f$  is obtained from the bandlimited function  $\mathcal{F}^{-1}F$  by time-warping with  $\eta^{-1}$ .

*Proof.* The proof follows easily, once we have identified the spectral measure and diagonalized  $B_p$ . The eigenfunctions  $-ip(x) \frac{d}{dx} \Phi(\lambda, x) = \lambda \Phi(\lambda, x)$  are given explicitly as

$$\Phi(\lambda, x) = e^{i\lambda \int_0^x \frac{dt}{p(t)}} = e^{i\lambda\eta(x)},$$

and the corresponding spectral transform is given by

$$U_p f(\lambda) = \int_{\mathbb{R}} f(x) \overline{\Phi(\lambda, x)} \frac{dx}{p(x)} = \int_{\mathbb{R}} f(x) e^{-i\lambda\eta(x)} \frac{dx}{p(x)}.$$

Using the substitution  $y = \eta(x)$ ,  $dy = \eta'(x)dx = \frac{dx}{p(x)}$ , we find that

$$U_p f(x) = \int_{\mathbb{R}} f(\eta^{-1}(y)) e^{-i\lambda y} dy = \mathcal{F}(f \circ \eta^{-1})(\lambda).$$

It is now easy to verify that  $U_p$  is unitary from  $L^2(\mathbb{R}, \frac{dx}{p(x)})$  onto  $L^2(\mathbb{R})$  and that  $U_p$  diagonalizes  $B_p$ , i.e.,  $U_p B_p U_p^* F(\lambda) = \lambda F(\lambda)$ . The inverse  $U_p^{-1} = U_p^* : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}, \frac{dx}{p(x)})$  is then given by

$$U_p^* F(x) = \int_{\mathbb{R}} F(\lambda) \Phi(\lambda, x) d\lambda = \int_{\mathbb{R}} F(\lambda) e^{i\lambda\eta(x)} d\lambda.$$

or  $U_p^* F = (\mathcal{F}^{-1} F) \circ \eta$ . The spectral projection of  $\Lambda$  is then  $c_{\Lambda}(B_p)f = U_p^* c_{\Lambda} U_p f$ . Consequently, every  $f \in PW_{\Lambda}(B_p)$  is given by

$$f(x) = \int_{\Lambda} F(\lambda) e^{i\lambda\eta(x)} d\lambda = (\mathcal{F}^{-1} F)(\eta(x))$$

for some  $F \in L^2(\Lambda)$ , as claimed.  $\square$

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